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Alexandre Martin

Topologie et géométrie des complexes de groupes à courbure négative ou nulle

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Goulnara Arzhantseva, examinateur
François Dahmani, examinateur
Thomas Delzant, directeur de thèse
Frédéric Haglund, rapporteur
Tadeusz Januszkiewicz, rapporteur
Frédéric Paulin, examinateur

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[1] Une chose est sûre, avec une telle force de persuasion, la théorie géométrique des groupes est définiti-

jury la personne qui tire les ficelles dans l'ombre depuis le début…

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[2] et une petite pensée émue également pour l'École Doctorale, ses formations absurdes et ses exigences farfelues (un exemplaire CD de la thèse, sérieusement ? !)

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Introduction

La théorie géométrique des groupes est la branche des mathématiques qui s'attache à étudier un groupe en le réalisant comme groupe de symétries d'un espace dont on juge la géométrie intéressante. La géométrie de l'espace sous-jacent ou les propriétés algébriques et dynamiques de l'action sont alors utilisées pour extraire des informations sur le groupe en question. Une telle approche présente deux leviers sur lesquels jouer.

D'un côté, on peut chercher à faire agir le groupe sur un espace dont la topologie ou la géométrie est particulièrement simple, avec potentiellement des stabilisateurs infinis. Un exemple illustrant une telle approche est donné par la théorie de Bass-Serre [43], c'est-à-dire l'étude des groupes qui agissent de manière non triviale sur des arbres simpliciaux. Ces groupes sont exactement les groupes qui se décomposent comme groupes fondamentaux de graphes de groupes. Cette théorie s'est révélée fructueuse pour démontrer (ou redémontrer) diverses propriétés algébriques de groupes. Citons par exemple :

- la liberté des sous-groupes discrets sans torsion de $SL_2(\mathbb{Q}_p)$ (voir [43]),
- le théorème de combinaison de Bestvina-Feighn [3], qui permet notamment de démontrer l'hyperbolicité des groupes fondamentaux de variétés de dimension 3 obtenues comme *mapping torus* au dessus d'un difféomorphisme pseudo-Anosov d'une surface hyperbolique,
- l'hyperbolicité relative des groupes limites par Dahmani [14].

À l'inverse, on peut chercher à créer une action avec des stabilisateurs finis, et tenter d'extraire de la géométrie de l'espace sous-jacent des informations algébriques sur le groupe en question. Un cas particulier est le cas des actions géométriques, c'est à dire propres et cocompactes, sur des espaces de dimension arbitraire. Dans un tel cas, le groupe devient quasi-isométrique à l'espace sur lequel il agit, et on a alors à disposition divers outils topologiques et géométriques pour étudier le groupe. Un point culminant de cette approche est sans conteste la théorie des groupes hyperboliques, introduite et développée par Gromov [23]. Ces groupes sont définis à partir d'une condition de finesse sur les triangles géodésiques de leurs graphes de Cayley, condition qui à elle seule a de nombreuses conséquences algébriques, dynamiques et algorithmiques. Cette classe de groupes est par ailleurs extrêmement vaste. Citons en exemple les groupes à petite simplification (voir [23]), les groupes obtenus par Davis et Januszkiewicz par hyperbolisation de complexes simpliciaux [15], les groupes 7-systoliques [33] et les réseaux uniformes dans les groupes de Lie semisimples de rang 1 (voir [12]).

Un autre exemple illustrant cette approche est le cas des groupes cubulables au sens de Wise [46], c'est à dire des groupes qui agissent de manière géométrique sur un complexe cubique CAT(0). Dans un tel cas, une condition de nature géométrique (typiquement, une action convenable sur un espace à murs) a des conséquences algébriques remarquables. Par exemple, le groupe fondamental d'un complexe cubique spécial compact est linéaire sur les entiers, résiduellement fini et séparable sur ses sous-groupes quasiconvexes [29].

Le présent travail se place dans la situation intermédiaire d'un groupe agissant de manière cocompacte, mais non nécessairement propre, sur un complexe de dimension quelconque à la géométrie contrôlée. Pour illustrer ce cas de figure, citons l'exemple du groupe modulaire d'une surface agissant sur son complexe des courbes. Masur et Minsky ont montré que ce complexe est hyperbolique [35], et Bowditch a montré l'acylindricité de l'action [7]. Pour autant, le groupe modulaire d'une surface n'est fortement hyperbolique relativement à aucune famille de sous-groupes [1]. Dans un autre ordre d'idée, Sageev a montré dans [41] comment l'existence d'un sous-groupe de codimension 1 peut entraîner l'existence d'une action cocompacte sur un complexe cubique CAT(0).

Dans de telles configurations, un problème naturel est de déterminer quelles propriétés du groupe proviennent des propriétés analogues pour ses stabilisateurs de faces. Plus précisément, le problème général suivant est le fil conducteur de toute cette thèse :

Problème de combinaison : Considérons un groupe G agissant cocompactement sur un complexe simplicial simplement connexe, et tel que chaque stabilisateur de simplexe vérifie une propriété $\mathcal{P}$ donnée. Existe-t-il des conditions sur la dynamique de l'action, sur la géométrie de l'espace, et sur les propriétés algébriques des stabilisateurs et de leurs inclusions, qui assurent que le groupe G satisfait lui aussi la propriété $\mathcal{P}$?

Notre étude permet une approche géométrique de groupes qui n'agissent pas de manière non triviale sur des arbres et ne jouissent pas d'une géométrie aussi riche que celle des groupes à courbure négative ou nulle, prise ici au sens large : groupes hyperboliques, CAT(0), ou encore systoliques. La théorie des actions de groupes sur des arbres simpliciaux trouve sa généralisation naturelle dans la théorie des complexes de groupes développée par Gersten-Stallings [44], Corson [13] et Haefliger [26]. Tout comme dans le cas de la théorie de Bass-Serre, les intérêts sont doubles. On peut d'un côté chercher à étudier un groupe en le faisant apparaître comme groupe fondamental d'un complexe de groupes dont on comprend les stabilisateurs et la géométrie. De l'autre, la théorie des complexes de groupes fournit de nouveaux exemples de groupes. Citons ici l'exemple des groupes de Coxeter hyperboliques de dimension cohomologique virtuelle arbitraire obtenus par Januszkiewicz et Świątkowski [32] à partir de complexes systoliques de groupes finis.

Résultats.

Cette thèse se place dans le cas d'un complexe de groupes à courbure négative ou nulle. Étant donné un tel complexe, on cherche à obtenir des propriétés de son groupe fondamental à partir des propriétés analogues pour ses groupes locaux. Les propriétés étudiées ici sont de trois types :

- existence d'un modèle cocompact d'espace classifiant pour les actions propres,
- existence d'un bord au sens de Bestvina,
- hyperbolicité.

Constructions d'espaces classifiants.

On s'intéresse en premier lieu à l'existence d'un modèle cocompact d'espace classifiant pour les actions propres d'un groupe G. Rappelons qu'un tel espace est un CW-complexe contractile muni d'une action propre et cocompacte de G, avec une condition sur les ensembles de points fixes des sous-groupes de G (voir I.4.1 pour une définition précise). Etant donné un complexe fini de groupes $G(\mathcal{Y})$ de groupe fondamental G, on cherche à construire un modèle cocompact d'espace classifiant pour G à partir de structures analogues pour ses groupes locaux. Dans le cas de la théorie de Bass-Serre, Scott et Wall [42] associent à un graphe fini de groupes une notion de graphes d'espaces qui leur permet de construire un espace d'Einlenberg-Mc Lane pour G. Un tel espace est un CW-complexe dont le revêtement universel est précisément un modèle cocompact d'espace classifiant pour les actions libres de G. Dans le cas plus général des complexes développables de groupes de dimension arbitraire, nous définissons de manière analogue une notion de complexe d'espaces compatible avec un complexe de groupes (voir II.2.1). Cela nous permet de construire un modèle cocompact d'espace classifiant pour les actions propres de G comme complexe d'espaces classifiants au dessus du revêtement universel de $G(\mathcal{Y})$. Nous démontrons le théorème suivant :

Théorème 1 : Soit $G(\mathcal{Y})$ un complexe développable de groupes au dessus d'un complexe simplicial fini Y, de revêtement universel contractile. S'il existe un complexe d'espaces classifiants compatible avec $G(\mathcal{Y})$, alors le groupe fondamental de $G(\mathcal{Y})$ admet un modèle cocompact d'espace classifiant pour les actions propres.

Comme exemple d'une telle construction, nous présentons la construction d'espaces classifiants pour les groupes à petite simplification sur un graphe fini de groupes. Ces groupes, qui généralisent la théorie ordinaire de la petite simplification, fournissent une classe intéressante de groupes qui peuvent ne pas agir de manière non triviale sur un arbre (voir par exemple les groupes hyperboliques étudiés par Delzant et Papasoglu [17]), mais agissent de manière cocompacte sur un complexe CAT(0) de dimension 2. Dans le cas de la petite simplification ordinaire, un espace classifiant pour les actions propres de G est obtenu comme revêtement universel du 2-complexe de Cayley obtenu à partir d'un bouquet de cercles en recollant un orbi-disque pour chaque relation de la présentation de G. De manière analogue, étant donné un graphe de groupes $G(\Gamma)$ et un quotient G à petite simplification métrique $C''(1/6)$, nous réalisons $G/\ll \mathcal{R} \gg$ comme groupe fondamental d'un complexe de groupes obtenu à partir du graphe de groupes $G(\Gamma)$ en lui recollant une collection d'orbi-disques. Nous prouvons ainsi le théorème suivant :

Théorème 2 : Soit $G(\Gamma)$ un graphe de groupes au dessus d'un graphe fini Γ, tel qu'il existe un graphe d'espaces classifiants compatible avec $G(\Gamma)$. Soit G le groupe fondamental de ce complexe de groupes et soit $\mathcal{R}$ un ensemble fini d'éléments de G qui agissent de manière hyperbolique sur l'arbre de Bass-Serre de $G(\Gamma)$. Si G n'admet pas d'élément non trivial fixant une droite et si $\mathcal{R}$ satisfait la condition de petite simplification métrique $C''(1/6)$, alors $G/\ll \mathcal{R} \gg$ possède un modèle cocompact d'espace classifiant pour les actions propres.

Donnons ici quelques détails sur les idées qui mènent à la construction. Le sous-groupe normal $\ll \mathcal{R} \gg$ agit sur l'arbre de Bass-Serre du graphe de groupes $G(\mathcal{Y})$, et les éléments de $\mathcal{R}$ agissent de manière hyperbolique, ce qui fournit une famille d'axes stable sous l'action de G. Nous utilisons cette famille pour construire le coned-off space $\widehat{T}$. L'espace quotient $\widehat{T}/\ll \mathcal{R} \gg$ se voit donc muni d'une action cocompacte de $G/\ll \mathcal{R} \gg$. Ce 2-complexe est le candidat naturel pour être un espace contractile avec une action cocompacte de $G/\ll \mathcal{R} \gg$. Toutefois, le fait que deux axes distincts puissent avoir plus d'une arête en commun rend difficile la question de la contractibilité. Pour parer à cette difficulté, nous suivons une idée de Gromov [25] et identifions certaines portions de ce 2-complexe ; on montre ensuite que le 2-complexe obtenu est localement CAT(0) grâce au critère de Gromov sur les links de sommets (voir I.2.10). Cependant, de manière à pouvoir construire un modèle cocompact d'espace classifiant pour $G/\ll \mathcal{R} \gg$ via les complexes d'espaces, il nous faut une compréhension fine des stabilisateurs, ce qui s'avère être une tâche ardue. Pour éviter cet écueil, nous changeons notre point de vue et construisons directement le complexe de groupes escompté, en utilisant des outils issus de la théorie des orbi-espaces introduite par Haefliger [26]. Nous prouvons que ce complexe de groupes est à courbure négative ou nulle, donc développable, et admet le groupe quotient $G/\ll \mathcal{R} \gg$ comme groupe fondamental. Une fois ce premier complexe de groupes défini, nous en construisons un second dont la combinatoire plus simple nous permet de lui associer un complexe d'espaces compatible. En appliquant les résultats précédents, on en déduit donc l'existence d'un modèle cocompact d'espace classifiant pour les actions propres de $G/\ll \mathcal{R} \gg$.

Constructions de bords de Bestvina.

Nous nous intéressons dans un deuxième temps à des compactifications des espaces classifiants construits ci-dessus. Dans [2], Bestvina définit une notion de bord de groupe qui est intéressante du point de vue de la géométrie des groupes et de la topologie géométrique. Par exemple, l'homologie d'un tel bord détermine la cohomologie à coefficients dans l'anneau du groupe. Farrell et Lafont [20] démontrent la conjecture de Novikov pour un groupe admettant une version équivariante de la notion de bord de Bestvina, notion qu'ils appellent $E\mathcal{Z}$-structure. La conjecture de Novikov étant un outil essentiel pour classifier des variétés ayant le même groupe fondamental à homéomorphisme près, la recherche de tels bords devient un problème naturel.

L'existence d'$E\mathcal{Z}$-structures, et sa généralisation aux groupes avec torsion, est connue pour les groupes admettant un espace classifiant dont la géométrie est à courbure négative ou nulle au sens large. Pour un groupe admettant une action géométrique sur un espace CAT(0), une telle compactification est obtenue en rajoutant à l'espace CAT(0) son bord visuel. Dans le cas d'un groupe hyperbolique sans torsion, un classifiant est donné par un complexe de Rips adéquat (voir I.3.10), et une $E\mathcal{Z}$-structure est obtenue en le compactifiant à l'aide du bord de Gromov du groupe [4]. Ce résultat est étendu au cas d'un groupe hyperbolique avec torsion dans [36]. L'existence d'une telle structure est également connue pour les groupes systoliques, introduits par Januszkiewicz et Swiatkowski [33] et indépendament par Haglund [28], d'après les résultats de Osajda-Przytycki [38].

Dans le cas d'un complexe de groupes à courbure négative ou nulle, nous donnons des conditions sous lesquelles il est possible d'amalgamer les différents bords en présence (bords de stabilisateurs et bord visuel du revêtement universel) pour obtenir un bord de Bestvina pour le groupe fondamental du complexe de groupes. Ces conditions sont de deux types :

- **Dynamique de l'action :** Nous nous restreignons au cas des actions *acylindriques*, c'est à dire les actions pour lesquelles il existe une borne uniforme sur le diamètre d'un sous-complexe stabilisé par un sous-groupe infini.
- **Propriétés algébriques et dynamiques des inclusions de stabilisateurs :** Le cas typique qui sera étudié est le cas d'un sous-groupe quasiconvexe d'un groupe hyperbolique.

Le théorème général de combinaison que nous démontrons est un peu trop technique pour être énoncé ici (voir IV.0.4). Il possède néanmoins le cas particulier suivant :

Théorème 3 : Soit $G(\mathcal{Y})$ un complexe de groupes simple à courbure négative ou nulle au dessus d'un M_κ-complexe fini Y ($\kappa \leq 0$), de groupe fondamental G et de revêtement universel X. Supposons que :

- l'action de G sur X est acylindrique,
- les groupes locaux G_σ sont hyperboliques et les injections $G_\sigma \hookrightarrow G_{\sigma'}$ sont des plongements quasiconvexes.

Alors G admet une $E\mathcal{Z}$-structure. De plus, on dispose d'une description explicite du bord de Bestvina.

La technique pour construire un tel bord suit une construction de Dahmani, utilisée dans [14] pour des graphes de groupes relativement hyperboliques.

Un théorème de combinaison pour les groupes hyperboliques.

Finalement, nous nous intéressons à une conséquence géométrique d'une telle construction. Partant d'un complexe de groupes à courbure négative ou nulle dont les groupes locaux sont hyperboliques, nous donnons des conditions qui assure l'hyperbolicité de son groupe fondamental. Plus précisément, nous prouvons le théorème suivant :

Théorème 4 : Sous les hypothèses du théorème 3, et si de plus X est hyperbolique, alors G est hyperbolique. De plus, les groupes locaux G_σ se plongent comme sous-groupes quasiconvexes de G.

Dans le cas d'un graphe de groupes, un tel résultat est une conséquence du théorème de combinaison de Bestvina-Feighn [3]. Un autre résultat de combinaison est connu dans le cas de groupes locaux commensurables [37].

Pour démontrer ce résultat, nous suivons la stratégie de Dahmani et étudions la dynamique de G sur le bord de Bestvina construit grâce aux résultats précédents. Nous démontrons que G est un groupe de convergence uniforme sur ce bord, ce qui implique le résultat par un théorème de caractérisation topologique de l'hyperbolicité dû à Bowditch [5].

Perspectives

Le travail effectué dans cette thèse est une première étape pour étudier des groupes à travers leurs actions cocompactes sur des espaces à courbure négative en un sens large : espaces CAT(0), hyperboliques, systoliques. Partant de là, il y a trois directions naturelles :

- Généraliser les théorèmes exposés à des actions plus générales pour obtenir des informations sur une classe plus large de groupes ;
- Appliquer ces résultats pour obtenir des résultats concrets sur différentes classes de groupes.
- Généraliser cette approche pour étudier des problèmes de combinaison pour d'autres types de propriétés d'un groupe.

Nous détaillons ici quelques pistes pour chacune de ces directions possibles.

Généralisation des résultats précédents. Un exemple pour lequel on souhaiterait appliquer un théorème de combinaison pour les bords de groupes est le cas du groupe modulaire d'une surface, agissant sur le complexe des courbes ou des arcs. Si les propriétés asymptotiques de ce groupe ont fait l'objet de nombreuses études (voir par exemple [31]), l'existence d'un bord de Bestvina pour ce groupe reste à ce jour inconnue.

Une première étape serait d'autoriser d'autres types de géométrie. Dans cette thèse, nous nous sommes concentrés sur le cas de groupes agissant sur des espaces CAT(0), espaces dont la richesse des géodésiques permettaient certaines constructions topologiques. Il serait intéressant de généraliser cette approche pour des espaces à la géométrie plus combinatoire (en particulier pour des complexes systoliques).

Une deuxième étape serait d'autoriser des actions plus générales. Nous nous sommes restreints ici au cas d'actions acylindriques avec des inclusions de stabilisateurs modelées sur le cas des sous-groupes quasi-convexes de groupes hyperboliques. Bien qu'il semble hors de portée pour l'instant de généraliser cette approche en relâchant complètement ces conditions, un premier cas qui semble riche d'enseignement serait le cas d'un produit amalgamé ou d'une extension HNN au dessus d'un sous-groupe quelconque (voir par exemple les constructions de bords pour les groupes de Baumslag-Solitar par Bestvina [2]).

Actions cocompactes sur des complexes cubiques CAT(0). Comme énoncé plus haut, on sait depuis les travaux de Sageev que l'existence d'un sous-groupe de codimension 1 d'un groupe G implique dans de nombreux cas l'existence d'une action cocompacte de G sur un complexe cubique CAT(0). Jusqu'à maintenant de nombreux travaux ont été effectués pour trouver suffisamment de tels sous-groupes afin d'obtenir une action qui soit également propre. Il serait néanmoins intéressant d'étudier si l'existence d'un seul tel sous-groupe H permet d'obtenir des informations sur G, via les théorèmes de combinaison obtenus dans cette thèse. En particulier, on peut se demander s'il existe des conditions naturelles sur la paire (G, H) pour que l'hyperbolicité de H implique l'hyperbolicité de G.

De nouvelles propriétés à "combiner". Si dans cette thèse nous nous sommes concentrés sur des propriétés de nature asymptotique (existence d'un espace classifiant et d'un bord), il est naturel de vouloir adopter cette approche pour d'autre propriétés d'un groupe. Citons-en ici quelques unes.

Les théorèmes de combinaison étudiés ici permettent d'étudier des groupes qui n'admettent aucune action non triviale sur des arbres simpliciaux. On peut naturellement se demander s'il est possible d'utiliser ces résultats pour créer des groupes hyperboliques ayant la propriété (T).

Une autre condition géométrique qui s'avère être riche de conséquences algébriques est la cubulation d'un groupe. En particulier, les résultats annoncés récemment par Agol ont des conséquences extrêmement fortes sur les groupes hyperboliques cubulables. Le cas des complexes de groupes cubulables semble donc être un problème qui fait naturellement suite au théorème des produits amalgamés de Haglund et Wise [30].

Enfin, dans une direction plus topologique, un problème intéressant serait d'obtenir un théorème de combinaison pour les groupes de Whitehead. Tout comme la conjecture de Novikov, le calcul de groupes de Whitehead est un outil important en topologie des variétés de grande dimension pour classifier des variétés à homéomorphisme près. Dans cette direction, l'un des seuls résultats de combinaison est le calcul par Waldhausen du groupe de Whitehead d'un produit amalgamé [45]. Il serait intéressant de comprendre comment la géométrie du groupe (et plus particulièrement son action sur l'arbre de Bass-Serre associé) apparaît en filigranes dans la preuve. On pourrait alors espérer généraliser ce calcul au cas des complexes cubiques de groupes.

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Chapter I

An overview of geometric group theory.

The aim of this first chapter is to gather the background material that will be used throughout this thesis, as well as fixing notations.

I.1 CAT(0) geometry.

We present here basic definitions and results about CAT(0) spaces and CAT(0) simplicial complexes. The standard reference for such spaces is [9].

I.1.1 CAT(0) metric spaces.

Definition I.1.1 (model space [9]). Given a real number κ, we define M^n_κ as the following metric space:

- if $\kappa = 0$ then M^n_0 is the n-dimensional Euclidean space;
- if $\kappa > 0$ then M^n_κ is the n-dimensional sphere with its usual spherical metric multiplied by a factor $1/\sqrt{\kappa}$;
- if $\kappa < 0$ then M^n_κ is the n-dimensional real hyperbolic space with its usual hyperbolic metric multiplied by a factor $1/\sqrt{-\kappa}$.

Note that if $n \geq 2$, then M^n_κ is the simply-connected Riemannian manifold of curvature κ.

Definition I.1.2 (geodesic segment, geodesic ray, geodesic space). Let X be a geodesic metric space and let $\gamma : [0, T] \to X$ (resp. $\gamma : [0, \infty) \to X$) be a continuous function. We say that γ *parametrises* a geodesic segment (resp. a geodesic ray) of X if for every t, t', we have $d\big(\gamma(t),\gamma(t')\big) = |t-t'|$. The image of γ is called a *geodesic segment* (resp. a *geodesic ray*) of X. Given a geodesic segment between two points $x, y \in X$, we denote it by $[x,y]$.

We say that a metric space X is *geodesic* if every pair of points of X is joined by a geodesic segment.

A *geodesic triangle* between points x, y, z of X is the reunion of a geodesic segment from x to y, from y to z, and from z to x.

Definition I.1.3 (comparison triangle)**.** Let X be a geodesic metric space, x, y, z points of X and κ a real number. Let $\Delta = \Delta([x,y],[y,z],[z,x])$ be a geodesic triangle. If $\kappa > 0$, further assume that $d(x,y) + d(y,z) + d(z,x) < 2\pi/\sqrt{\kappa}$ (the latter constant being the diameter of M^2_κ).

A *comparison triangle* in M^2_κ is the unique (up to isometry) geodesic triangle $\Delta' = \Delta'([x',y'],[y',z'],[z',x'])$ of M^2_κ such that $d(x',y') = d(x,y)$, $d(y',z') = d(y,z)$ and $d(z,'x') = d(z,x)$.

Let $u \in \Delta$ be a point of the geodesic segment $[x,y]$. The associated *comparison point* in Δ' is the unique point u' of $[x',y']$ such that $d(x',u') = d(x,u)$. Comparison points for points in Δ are defined in the same way.

Definition I.1.4 (CAT(κ) space)**.** Let (X,d) be a geodesic metric space and κ a real number. The space X is a CAT(κ) *space* if for all points x, y, z of X (subject to the additional requirement that $d(x,y) + d(y,z) + d(z,x) < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$), for every geodesic triangle $\Delta = \Delta([x,y],[y,z],[z,x])$, the associated comparison triangle $\Delta' = \Delta'([x',y'],[y',z'],[z',x'])$ of M^2_κ is such that for every two points u, v of Δ and their comparison points u', v' of Δ', we have $d(u,v) \leq d(u',v')$.

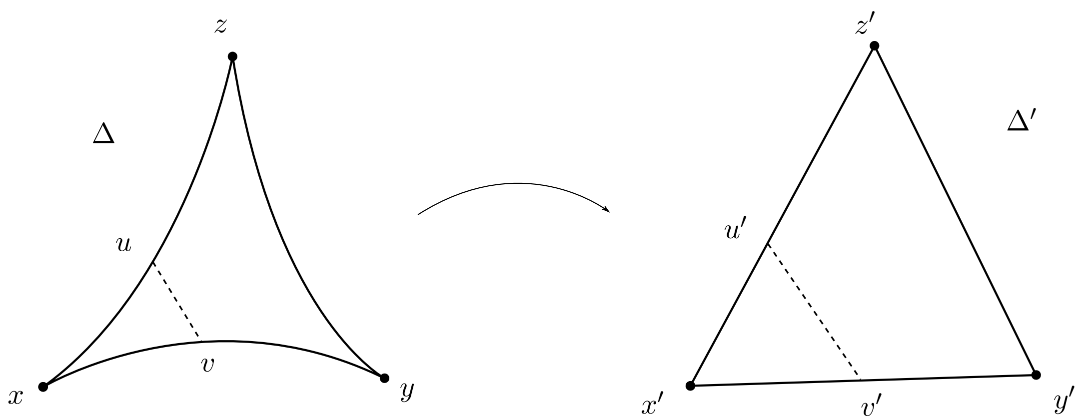

Figure I.1 - The CAT(0) condition.

The CAT(0) property has many useful consequences.

Proposition I.1.5 (II.1.1.4, II.2.2.2, II.4.4.5 of [9])**.** *Let X be a CAT(0) space. Then:*

- *The distance is convex, that is, for all $\gamma : [0,T] \to X$ and $\gamma' : [0,T'] \to X$ parametrising geodesic segments of X, the map $t \mapsto d\big(\gamma(t), \gamma'(t)\big)$ is convex.*

- *For every two points x, y of X, there is a unique geodesic segment joining x to y, and this geodesic varies continuously with its endpoints.*

- *(Local geodesics are geodesics) A map $\gamma : I \to X$ parametrises a geodesic if and only if each of its restrictions to a sub-interval of I does itself parametrise a geodesic of X.* □

Theorem I.1.6 (II.2.2.8 of [9])**.** *If X is a CAT(0) space and G a finite group acting by isometries on X, then the fixed-point set of G is a non-empty convex subset of X. In particular, there exists a point of X fixed by G.* □

I.1.2 The boundary at infinity of a complete CAT(0) space.

Let X be a complete CAT(0) space and x_0 a basepoint.

Definition I.1.7 (Boundary at infinity of a CAT(0) space)**.** Two geodesic rays $c, c' : [0,\infty) \to X$ are called *asymptotic* if there exists a constant k such that $d\big(c(t), c'(t)\big) \leq k$ for every t. The *boundary at infinity* of X, denoted $\partial_\infty X$ or simply ∂X when no confusion is possible, is the set of equivalence classes of geodesic rays, two rays being equivalent if they are asymptotic. The union $\overline{X} = X \cup \partial X$ is called the *bordification of* X. Let x_0 be a basepoint of X. The boundary ∂X identifies with the set of geodesic rays issuing from x_0. For $r > 0$ and for a point $x \in \overline{X}$ which is not in the open r-ball centred at x_0, we denote by $\pi_r(x)$ the unique point of the geodesic from x_0 to x which is at distance r from x_0.

We now define a topology on the bordification of a CAT(0) space.

Definition I.1.8 (Topology of the bordification)**.** We define a topology on the bordification of X as follows. The space X is an open subset of $\overline{X}$. For a point $\eta \in \partial X$, a basis of neighbourhoods of η in $\overline{X}$ is given by the family of subsets

$$V_{r,\varepsilon}(\eta) = \{x \in \overline{X} : d\big(\pi_r(x), \pi_r(\eta)\big) < \varepsilon\}, \quad r, \varepsilon > 0.$$

By embedding $\overline{X}$ in an appropriate (metrisable) function space, one has the following:

Proposition I.1.9 (Metrisability of the bordification, II.8.8.13 of [9])**.** *Endowed with that topology, the bordification $\overline{X}$ is a metrisable space. If in addition X is locally compact, then $\overline{X}$ is compact.* □

I.2 Simplicial complexes and their CAT(0) geometry.

I.2.1 The geometry of M_κ-simplical complexes.

Definition I.2.1 (star, link of a simplicial complex)**.** Let X be a simplicial complex and v a vertex of X. The *star* of v, denoted $\mathrm{St}(v, X)$ or simply $\mathrm{St}(v)$ when no confusion is possible, is the subcomplex of X which is the union of the simplices that contain v. The *open star* of v, denoted $\mathrm{st}(v, X)$ or $\mathrm{st}(v)$ when no confusion is possible, is the union of the open simplices whose closure contains v. The *link* of v, denoted $\mathrm{lk}(v, X)$, is the subcomplex $\mathrm{St}(v) \setminus \mathrm{st}(v)$. Equivalently, it is the subcomplex of X which is the union of the simplices of $\mathrm{St}(v)$ that do not contain v.

Definition I.2.2 (Simplicial neighbourhood)**.** Let X be a simplicial complex and K be a subcomplex of X. The union of the closed simplices that meet K is called the *closed simplicial neighbourhood* of K, and denoted $\overline{\mathrm{N}}(K)$. The union of the open simplices whose closure meets K is called the *open simplicial neighbourhood* of K, and denoted $\mathrm{N}(K)$.

Since the present thesis focuses on nonproper group actions, the simplicial complexes considered herein will not be locally finite in general. Endowing such spaces with a satisfying topology turns out to be a non trivial problem. In [8], Bridson introduced a class of spaces that is suitable for a geometric approach.

Definition I.2.3 (M_κ-simplicial complexes, [8])**.** Let κ be a real number. A simplicial complex X is called a M_κ-simplicial complex if it satisfies the following two conditions:

- Each simplex of X is modeled after a geodesic simplex in some M_κ^n, that is, each simplex σ comes with a bijection f_σ from σ to a simplex of M_κ^n of the same dimension.
- If σ and σ' are two simplices of X sharing a common face τ, the composition $f_{\sigma'} \circ f_\sigma^{-1}$ is an isometry from $f_\sigma(\tau)$ to $f_{\sigma'}(\tau)$.

Simplicial M_κ-complexes with $\kappa = 0$ (resp. $\kappa = 1$, resp $\kappa = -1$) are called piecewise Euclidean (resp. piecewise spherical, resp. piecewise hyperbolic) complexes.

Given a M_κ-simplicial complex, it is always possible to consider the associated simplicial pseudometric, as described below.

Definition I.2.4 (simplicial pseudometric [9])**.** Let X be a M_κ-simplicial complex, x, y two points of X. An *m-string* from x to y is a sequence $\Sigma = (x_0, \dots, x_m)$ of points of X such that $x_0 = x, x_m = y$, and for each $i = 0, \dots, m-1$, there exists a simplex σ_i containing x_i and x_{i+1}. We define the *length* of Σ as

$$l(\Sigma) = \sum_{0 \le i \le m-1} d_{\sigma_i}(x_i, x_{i+1}),$$

where d_{σ_i} is the distance on σ_i induced by the M_κ-structure of X. The pseudometric on X is defined by

$$d(x, y) = \inf\{l(\Sigma) \mid \Sigma \text{ a string from } x \text{ to } y\}.$$

A fundamental result of Bridson's thesis is the following:

Theorem I.2.5 (Bridson [8])**.** *If X is a M_κ-simplicial complex with finitely many isometry types of simplices, then the simplicial pseudometric is a complete and geodesic metric.* □

Remark: Choosing different M_κ-structures, κ not fixed, yields bi-Lipschitz equivalent metrics [9, p. 128].

To some extent, the geometry of M_κ-simplicial complexes, $\kappa \leq 0$, parallels the geometry of locally finite complexes. Here is an extremely useful illustration of such a similarity:

Proposition I.2.6 (containment lemma, Bridson [8])**.** *Let X be a M_κ-simplicial complex, $\kappa \leq 0$, with finitely many isometry types of simplices. For every n there exists a constant k such that for every finite subcomplex $K \subset X$ containing at most n simplices, any geodesic path contained in the simplicial neighbourhood of K meets at most k simplices.* □

Corollary I.2.7 (Bridson [8])**.** *Let X be a M_κ-simplicial complex, $\kappa \leq 0$, with finitely many isometry types of simplices. For every n there exists a constant k such that every geodesic segment of X of length at most n meets at most k simplices.* □

Throughout this thesis, every simplicial complex will implicitely be given a structure of M_κ-complex, $\kappa \leq 0$, unless stated otherwise.

I.2.2 Simplicial complexes with a CAT(0) simplicial metric.

The CAT(0) condition is a global condition on the geometry of a metric space, making it particularly hard to check. In the case of a simply-connected simplicial complex endowed with its simplicial metric, the CAT(0) condition boils down to a local condition.

Definition I.2.8 (piecewise spherical metric on the link)**.** Let X be a M_κ-simplicial complex with finitely isometry types of simplices, and v a vertex of X. Let τ be a simplex of $\mathrm{lk}(v, X)$ and x, y two points of τ. Recall that this implies that there exists a simplex σ of X containing v and τ and such that τ does not contain v. We define the *angular distance* between x and y, denoted $\angle(x, y)$, as the angle at v (measured in σ) between the geodesic segments $[v, x]$ and $[v, y]$. This endows each simplex of $\mathrm{lk}(v, X)$ with a piecewise spherical metric, and endows $\mathrm{lk}(v, X)$ with a structure of piecewise spherical complex with finitely isometry types of simplices.

The *angular metric* on $\mathrm{lk}(v, X)$ is the associated simplicial metric.

Theorem I.2.9 (Gromov's criterion [23])**.** *Let X be a simply-connected M_κ-complex with finitely isometry types of simplices. Then X is CAT(κ) if and only if the link of every vertex of X is CAT(1) for the angular metric.* □

Checking this local condition turns out to be difficult in full generality. We present here a particular case, which will be used in this thesis, for which this condition is much simpler to check.

Corollary I.2.10 (Gromov's criterion [23])**.** *A 2-dimensional simply-connected M_κ-complex with finitely many isometry types of simplices is CAT(κ) if and only if an essential loop in the link of any vertex has length at least 2π for the associated angular metric.* □

Another case for which this local condition boils down to a much more combinatorial condition is the case of a finite-dimensional CAT(0) cube complex. Although this criterion will not be explicitly used in this thesis, it makes the class of CAT(0) cube complexes particularly suitable to create non-positively curved complexes of groups. We first recall a standard definition:

Definition I.2.11 (flag complex)**.** A simplicial complex is *flag* if every set of vertices pairwise connected by an edge spans a simplex.

Theorem I.2.12 (Gromov's criterion, [23])**.** *Let X be a simply-connected finite-dimensional cube complex. Then X is CAT(0) if and only if the link of each of its vertices is a flag complex.* □

I.3 Hyperbolic groups.

This section presents elementary material about hyperbolic groups. This class of groups, introduced by Gromov [23], has an extremely rich geometry and has been extensively studied during the past twenty years. For a much more in-depth discussion of these groups, we refer to [12].

I.3.1 Hyperbolic metric spaces and hyperbolic groups.

Let (X, d) be a metric space.

Definition I.3.1 (Gromov product)**.** We define the *Gromov product* of x and y at z as follows:

$$\langle x, y\rangle_z = \frac{1}{2}\big(d(x,z) + d(y,z) - d(x,y)\big).$$

Definition I.3.2 (δ-hyperbolic space)**.** Let p be a basepoint of X and $\delta > 0$. We say that (X, p) is *δ-hyperbolic* if for every $x, y \in X$:

$$\langle x, y\rangle_p \geq \min_{z \in X}\big(\langle x, z\rangle_p, \langle y, z\rangle_p\big) - \delta.$$

Changing the basepoint only changes the hyperbolicity constant. We thus have the following definition:

Definition I.3.3 (hyperbolic space)**.** We say that a metric space X is hyperbolic if there exists a point p and a constant $\delta > 0$ such that (X, p) is δ-hyperbolic.

In the case of geodesic metric spaces, there is a very visual alternative definition of hyperbolicity.

Proposition I.3.4 (thin triangles)**.** *Let X be a geodesic metric space. Then X is hyperbolic if and only if there exists $\delta > 0$ such that X has δ-thin geodesic triangles, that is, for every $x, y, z \in X$ and geodesic segments $[x, y], [x, z], [y, z]$, we have*

$$[x, y] \subset N_\delta\big([x, z]\big) \cup N_\delta\big([z, y]\big),$$

where $N_\delta(.)$ represents closed δ-neighbourhoods.

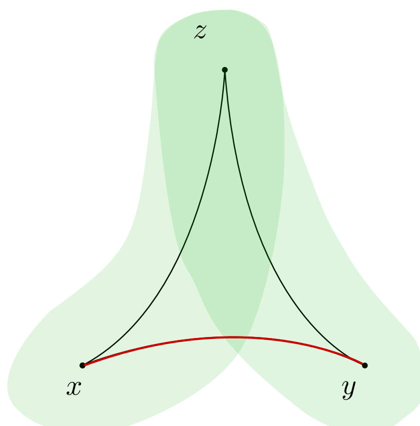

Figure I.2 - Thin triangles.

Hyperbolicity is a property preserved under a fundamental kind of application that we now describe:

Definition I.3.5 (quasi-isometric embedding, quasi-isometry)**.** Let $(X, d), (X', d')$ be two metric spaces, and $f : X \to X'$ a map. We say that f is a *quasi-isometric embedding* if there exists constant $\lambda \geq 1$ and $\varepsilon \geq 0$ such that for all $x, y \in X$,

$$\frac{1}{\lambda} d(x, y) - \varepsilon \leq d'(f(x), f(y)) \leq \lambda d(x, y) + \varepsilon.$$

If in addition there exists a constant $C \geq 0$ such that every point of X' is in the C-neighbourhood of the image of X, we say that f is a *quasi-isometry*. When there exists a quasi-isometry between two metric spaces, we say that they are *quasi-isometric*.

Proposition I.3.6 (III.2.2 of [12])**.** *Let X, X' be two quasi-isometric metric spaces. Then X is hyperbolic if and only if X' is.*

Definition I.3.7 (word metric)**.** Let G be a finitely generated group and S a finite generating set. For elements $g, g' \in G$, we set $d_S(g, g') = 0$ if $g = g'$, and we otherwise define $d_S(g, g')$ as the smallest positive integer n such that there exists elements $g_1, \ldots, g_n \in S \cup S^{-1}$ such that $g^{-1}g' = g_1 \ldots g_n$. The function d_S defines a metric on G called the *word metric* associated to the generating set S.

One goal of geometric group theory is to understand a finitely generated group through the associated metric space obtained by choosing a word metric. If S and S' are two finite generating sets, then (G, d_S) and $(G, d_{S'})$ are quasi-isometric, which makes the notion of a quasi-isometry of fundamental importance. This motivates the following definition:

Definition I.3.8. Let G be a finitely generated group. We say that G is hyperbolic if for one (hence every) word metric d_S, the metric space (G, d_S) is hyperbolic.

I.3.2 The Rips complex of a hyperbolic group.

Given a finitely generated group G with a finite generating set S, one wants to understand G by making it act on a metric space with an interesting topology or geometry. A standard space on which G acts by isometries is its Cayley graph with respect to a given generating set. We present here a fundamental construction that yields a contractible space on which G acts.

Definition I.3.9 (Rips complex)**.** Let G be finitely generated group and d_S a word metric associated to a finite generating set S. For an integer $d \geq 0$, we define a simplicial complex $P_d(G)$ in the following way. Vertices of $P_d(G)$ correspond to elements of G. Elements $g_0, \ldots, g_k \in G$ span a k-simplex if $d_S(g_i, g_j) \leq d$ for every $0 \leq i, j \leq k$.

We have the following fundamental theorem:

Theorem I.3.10 (Gromov-Rips, see V.2.2 of [12])**.** *Let G be a hyperbolic group with generating set S. For d large enough, the Rips complex $P_d(G, S)$ is contractible. Furthermore, G acts on the first barycentric subdivision of $P_d(G, S)$ by simplicial isomorphism without inversion, cocompactly and properly.* □

This theorem has many implications. Let us present a few of them.

Corollary I.3.11 (V.2.3, V.2.4 of [12])**.** *Let G be a hyperbolic group. Then:*

- *G is finitely presented,*
- *the cohomology groups $H^k(G, \mathbb{Q})$ vanish for k large enough.* □

I.3.3 The Gromov boundary of a hyperbolic group.

Let G be a hyperbolic group.

Definition I.3.12 (Gromov boundary of a hyperbolic space)**.** Let p be a basepoint of X. We say that a sequence (x_n) of points of X *goes to infinity* if $\langle x_n, x_m\rangle_p \underset{n,m}{\to} \infty$. Two sequences $(x_n), (x'_n)$ going to infinity are called equivalent, denoted $(x_n) \simeq (x'_n)$, if $\langle x_n, x'_n\rangle_p \to \infty$. We define the *Gromov boundary of* X, denoted $\partial_{\text{Gromov}} X$ or simply ∂X when no confusion is possible, as the set of equivalence classes of sequences of X going to infinity. For such an equivalence class η, we say that a sequence (x_n) *converges to* η if it is in the equivalence class η. The union $\overline{X} = X \cup \partial X$ is called the *bordification* of X.

Definition I.3.13. The Gromov product extends to $\overline{X}$ by the formula

$$\langle \eta, \eta'\rangle_p = \sup \liminf_{n,m\to\infty} \langle x_n, x'_m\rangle_p$$

where the supremum is taken over all the sequences $(x_n), (x'_n)$ of X such that (x_n) converges to η and (x'_n) converges to η'.

Definition I.3.14 (Topology of the bordification)**.** We define a topology on $X \cup \partial X$ as follows. For this topology, X is an open subset of $X \cup \partial X$. Moreover, for a point η of ∂X, a basis of neighbourhoods at η is given by the family of subsets

$$W_k(\eta) = \{\xi \in \overline{X} : \langle \xi, \eta\rangle_p \geq k\}, \quad k \geq 1.$$

Note that we did not indicate the dependance on the basepoint p. This is justified by the following observation:

Proposition I.3.15. *Let X, X' two hyperbolic metric spaces and $f : X \to X'$ a quasi-isometry. Then f extends to a homeomorphism from ∂X to $\partial X'$.* □

For a complete geodesic metric space X which is both hyperbolic and CAT(0), we have a priori two notions of a bordification, namely the ones obtained by adding the visual boundary or the Gromov boundary. In this case, the two notions coincide, as explained below. This is proven for proper spaces in [9, Prop. III.H.3.7], but the proof generalises to the following:

Proposition I.3.16. *Let X be a complete geodesic metric space which is both hyperbolic and CAT(0). The identity of X extends to a homeomorphism from $X \cup \partial_\infty X$ to $X \cup \partial_{\text{Gromov}} X$.* □

Definition I.3.17 (Gromov boundary of a hyperbolic group)**.** Let G be a hyperbolic group. The *Gromov boundary of* G is the Gromov boundary of the hyperbolic space G for any of its word metrics.

Note that the action of G on itself on the left extends to an action on ∂G.

Proposition I.3.18. *The Gromov boundary of a hyperbolic group is a compact metrisable space. The group G acts on its Gromov boundary by homeomorphisms.* □

I.3.4 Convergence groups and hyperbolicity.

Here we explain how the dynamics of a hyperbolic group on its Gromov boundary yields a topological characterisation of hyperbolicity.

Definition I.3.19 (convergence group)**.** A group G acting on a compact metrisable space M with more than two points is called a *convergence group* if, for every sequence (g_n) of elements of G, there exists two points ξ_+ and ξ_- in M and a subsequence $(g_{\varphi(n)})$, such that for any compact subspace $K \subset M \setminus \{\xi_-\}$, the sequence $(g_{\varphi(n)}K)$ of translates uniformly converges to ξ_+.

Since the inclusion of G in one its Rips complex is a quasi-isometry, ∂G is also the Gromov boundary of any of its Rips complexes.

Proposition I.3.20 (Freden [21])**.** *Let G be a hyperbolic group. Then G is a convergence group on $G \cup \partial G$. More generally, let $P_d(G,S)$ be some Rips complex of G. Then G is also a convergence group on $P_d(G,S) \cup \partial G$.* □

Definition I.3.21 (conical limit point)**.** Let G be a convergence group on a compact metrisable space M. A point ζ in M is called a *conical limit point* if there exists a sequence (g_n) of elements of G and two points $\xi_- \neq \xi_+$ in M, such that $g_n\zeta \to \xi_-$ and $g_n\zeta' \to \xi_+$ for every $\zeta' \neq \zeta$ in M. The group G is called a *uniform convergence group* on M if M consists only of conical limit points.

Theorem I.3.22 (Bowditch [5])**.** *Let G be a uniform convergence group on a compact metrisable space M with more than two points. Then G is hyperbolic and M is G-equivariantly homeomorphic to the Gromov boundary of G.* □

I.3.5 Quasiconvex subgroups of hyperbolic groups.

While arbitrary subgroups of a finitely generated group can be extremely wild, hyperbolic groups possess an important class of subgroups with a very controlled geometry.

Definition I.3.23 (quasiconvexity)**.** Let X be a geodesic metric space. A subset $Y \subset X$ is called *quasiconvex* if there exists $\alpha \geq 0$ such that every geodesic between two points of Y lies in the α-neighbourhood of Y.

Definition I.3.24 (quasiconvex subgroup with respect to a finite generating set)**.** Let G be a finitely generated group and S a finite generating set. A subgroup $H < G$ is said to be *quasiconvex with respect to S* if H is a quasiconvex subset of the Cayley graph of G associated to S.

In the case of a hyperbolic group, the notion of a quasiconvex subgroup does not depend on the choice of a finite generating set (see [12, Prop. 10.4.1]). In such a case, we simply speak of a *quasiconvex subgroup*. Here are a few properties of quasiconvex subgroups of a hyperbolic group (we refer to [12]).

Proposition I.3.25. *Let G be a hyperbolic group and H a quasiconvex subgroup. Then H is hyperbolic and the inclusion $H \hookrightarrow G$ is a quasi-isometric embedding.* □

Proposition I.3.26. *Let G be a hyperbolic group and H a quasiconvex subgroup. Then the inclusion $H \hookrightarrow G$ extends to an embedding $H \cup \partial H \hookrightarrow G \cup \partial G$. More generally, let S be a finite generating set of G and S' a finite generating set of H which is contained in S. Then the equivariant embedding of Rips complexes $P_d(H, S') \hookrightarrow P_d(G, S)$ naturally extends to an equivariant embedding $P_d(H, S') \cup \partial H \hookrightarrow P_d(G, S) \cup \partial G$.*

Definition I.3.27 (limit set)**.** Let G be a hyperbolic group and H a subgroup. The *limit set* of H is the set $\Lambda H = \overline{H} \cap \partial G$, where $\overline{H}$ denotes the closure of the set H, seen as a subspace of $G \cup \partial G$.

Theorem I.3.28 (Bowditch [6])**.** *Let G be a hyperbolic group and H a hyperbolic subgroup of G. Then H is quasiconvex if and only if the limit set ΛH is H-equivariantly homeomorphic to the Gromov boundary of H.* □

Lemma I.3.29. *Let G be a hyperbolic group, and H_1, H_2 two subgroups of G.*

- *Suppose that $H_1 \leq H_2$. If H_1 is quasiconvex in H_2, and H_2 is quasiconvex in G, then H_1 is quasiconvex in G. If both H_1 and H_2 are quasiconvex in G, then H_1 is quasiconvex in H_2.*
- *(Gromov [24, p.164]) Suppose that H_1, H_2 are quasiconvex subgroups of G. Then $H_1 \cap H_2$ is quasiconvex in G, and $\Lambda(H_1 \cap H_2) = \Lambda H_1 \cap \Lambda H_2$.* □

I.4 Classifying spaces and boundaries of groups.

Geometric group theory tries to understand a group through its actions on topological spaces. We present here a fundamental example of such a space.

Definition I.4.1 ((cocompact model of a) classifying space for proper actions)**.** Let G be a finitely generated group. A *cocompact model of a classifying space for proper actions of* G (or briefly a *classifying space for* G) is a contractible CW-complex EG with a properly discontinuous cocompact and cellular action of G, such that for every finite subgroup H of G, the fixed point set EG^H is nonempty and contractible.

Compactifications of such spaces are one of the main topics of this thesis. The original notion of $\mathcal{Z}$-structure is due to Bestvina [2]. A generalisation for groups with torsion was introduced by Dranishnikov [19]. Farrell and Lafont [20] studied an equivariant analogue, which they call an $E\mathcal{Z}$-structure.

Definition I.4.2 ($\mathcal{Z}$-structures, $E\mathcal{Z}$-structures)**.** Let G be a discrete group. A *$\mathcal{Z}$-structure* for G is a pair (Y, Z) of spaces such that:

- Y is a Euclidean retract, that is, a compact, contractible and locally contractible space with finite covering dimension,
- $Y \setminus Z$ is a classifying space for proper actions of G,
- Z is a $\mathcal{Z}$-set in Y, that is, Z is a closed subspace of Y such that for every open set U of Y, the inclusion $U \setminus Z \hookrightarrow U$ is a homotopy equivalence,
- Compact sets *fade at infinity*, that is, for every compact set K of $Y \setminus Z$, every point $z \in Z$ and every neighbourhood U of z in Y, there exists a subneighbourhood $V \subset U$ with the property that if a G-translate of K intersects V, then it is contained in U.

The pair (Y, Z) is called an *$E\mathcal{Z}$-structure* if in addition we have:

- The action of G on $Y \setminus Z$ continuously extends to Y.

The importance of such structures stems from the following theorem:

Theorem I.4.3 ([20]). *If G admits an $E\mathcal{Z}$-structure, then G satisfies the Novikov conjecture.* □

We now present a slightly stronger notion of boundary, which also has stronger implications for the Novikov conjecture.

Definition I.4.4. Let G be a group endowed with an $E\mathcal{Z}$-structure $(\overline{EG}, \partial G)$. We say that $(\overline{EG}, \partial G)$ is an *$E\mathcal{Z}$-structure in the sense of Carlsson-Pedersen* if in addition we have: For every finite group H of G, the fixed point set $\overline{EG}^H$ is nonempty and admits EG^H as a dense subset.

The importance of such finer structures comes from the following implication.

Theorem I.4.5 ([11], [40]). *If G admits an $E\mathcal{Z}$-structure in the sense of Carlsson-Pedersen, then G satisfies the generalised integral Novikov conjecture.* □

In the case of a hyperbolic group, there is a very explicit example of a classifying space for proper actions, namely the Rips complex (see [36]). Moreover, there is a natural notion of boundary, namely the Gromov boundary.

Theorem I.4.6 ([4], [36]). *Let G be a hyperbolic group and S a finite generating set of G. For d large enough, the Rips complex $P_d(G, S)$ is contractible and the topology on $P_d(G, S) \cup \partial G$ makes $(P_d(G, S) \cup \partial G, \partial G)$ an $E\mathcal{Z}$-structure in the sense of Carlsson-Pedersen for G.* □

I.5 Complexes of groups.

I.5.1 First definitions.

Graphs of groups are algebraic objects that were introduced by Serre [43] to encode group actions on trees. To every cocompact action without inversion of a group G on a simplicial tree, Bass-Serre theory associates a graph of groups structure on the quotient graph, called an *induced* graph of groups. Reciprocally, to every graph of groups one can associate an action of its fundamental group on a simplicial tree, the *Bass-Serre tree* of the graph of groups, with quotient the given simplicial graph. Moreover, the fundamental group of a graph of groups induced by the action of a group G on a simplicial tree T is isomorphic to G, and its universal cover is G-equivariantly isometric to T.

Thus, graphs of groups can be seen as encoding cocompact actions of groups on trees. If one wants to generalise that theory to higher dimensional complexes, one needs the theory of complexes of groups developed by Haefliger [26]. Haefliger defined a notion of complexes of groups over more general objects called *small categories without loops* (abbreviated *scwol*), a combinatorial generalisation of polyhedral complexes. Although in this thesis we will only deal with actions on simplicial complexes, we use the terminology of scwols to be coherent with the existing literature on complexes of groups. For a deeper treatment of the material covered in this paragraph and for the general theory of complexes of groups over scwols, we refer the reader to [9].

Definition I.5.1 (small category without loop)**.** A *small category without loop* (briefly a *scwol*) is a set $\mathcal{X}$ which is the disjoint union of a set $V(\mathcal{X})$ called the vertex set of $\mathcal{X}$, and a set $E(\mathcal{X})$ called the edge set of $\mathcal{X}$, together with maps

$$i : E(\mathcal{X}) \to V(\mathcal{X}) \text{ and } t : E(\mathcal{X}) \to V(\mathcal{X}).$$

For an edge $a \in E(\mathcal{X})$, $i(a)$ is called the initial vertex of a and $t(a)$ the terminal vertex of a.

Let $E^{(2)}(\mathcal{X})$ be the set of pairs $(a, b) \in E(\mathcal{X})$ such that $i(a) = t(b)$. A third map

$$E^{(2)}(\mathcal{X}) \to E(\mathcal{X})$$

is given that associates to such a pair (a, b) an edge ab called their composition (and a and b are said to be composable). These maps are required to satisfy the following conditions:

- For every $(a, b) \in E^{(2)}(\mathcal{X})$, we have $i(ab) = i(b)$ and $t(ab) = t(a)$;
- For every $a, b, c \in E(\mathcal{X})$ such that $t(a) = i(b)$ and $t(b) = i(c)$, we have $(ab)c = a(bc)$ (and the composition is simply denoted abc).
- For every $a \in E(\mathcal{X})$, we have $t(a) \neq i(a)$.

Definition I.5.2 (simplicial scwol associated to a simplicial complex)**.** If X is a simplicial complex, a scwol $\mathcal{X}$ is naturally associated to X in the following way:

- $V(\mathcal{X})$ is the set $S(X)$ of simplices of X,
- $E(\mathcal{X})$ is the set of pairs $(\sigma, \sigma') \in V(\mathcal{X})^2$ such that $\sigma \subset \sigma'$.
- For a pair $a = (\sigma, \sigma') \in E(\mathcal{X})$, we set $i(a) = \sigma'$ and $t(a) = \sigma$.
- For composable edges $a = (\sigma, \sigma')$ and $b = (\sigma', \sigma'')$, we set $ab = (\sigma, \sigma'')$.

We call $\mathcal{Y}$ the *simplicial scwol associated to* Y.

In what follows, we will often omit the distinction between a simplex σ of Y and the associated vertex of $\mathcal{Y}$.

Definition I.5.3 (Complex of groups [9])**.** Let $\mathcal{Y}$ be a scwol. A *complex of groups* $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ *over* Y is given by the following data:

- for each vertex σ of $\mathcal{Y}$, a group G_σ called the *local group* at σ,
- for each edge a of $\mathcal{Y}$, an injective homomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$,
- for each pair of composable edges (a, b) of $\mathcal{Y}$, a *twisting element* $g_{a,b} \in G_{t(a)}$,

with the following compatibility conditions:

- for every pair (a, b) of composable edges of $\mathcal{Y}$, we have
$$\mathrm{Ad}(g_{a,b})\psi_{ab} = \psi_a\psi_b,$$
where $\mathrm{Ad}(g_{a,b}) : g \mapsto g_{a,b} \cdot g \cdot g_{a,b}^{-1}$ is the conjugation by $g_{a,b}$ in $G_{t(a)}$;
- if (a, b) and (b, c) are pairs of composable edges of $\mathcal{Y}$, then the following cocycle condition holds:
$$\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}.$$

A complex of groups is called *simple* if all the twisting elements are trivial. If Y is a simplicial complex, a *complex of groups over* Y is a complex of groups over the associated simplicial scwol.

Definition I.5.4 (Morphism of complex of groups)**.** Let Y, Y' be simplicial complexes, $\mathcal{Y}$ (resp. $\mathcal{Y}'$) the associated simplicial scwols, $f : Y \to Y'$ a non-degenerate simplicial map (that is, the restriction of f to any simplex is a homeomorphism on its image) , and $G(\mathcal{Y})$ (resp. $G(\mathcal{Y}')$) a complex of groups over Y (resp. Y'). A *morphism* $F = (F_\sigma, F(a)) : G(\mathcal{Y}) \to G(\mathcal{Y}')$ over f consists of the following:

- for each vertex σ of $\mathcal{Y}$, a homomorphism $F_\sigma : G_\sigma \to G_{f(\sigma)}$,
- for each edge a of $\mathcal{Y}$, an element $F(a) \in G_{t(f(a))}$ such that
  1. for every pair (a, b) of composable edges of $\mathcal{Y}$, we have
  $$\mathrm{Ad}(F(a))\psi_{f(a)}F_{i(a)} = F_{t(a)}\psi_a,$$
  2. if (a, b) and (b, c) are pairs of composable edges of $\mathcal{Y}$, we have
  $$F_{t(a)}(g_{a,b})F(ab) = F(a)\psi_{f(a)}(F(b))g_{f(a),f(b)}.$$

If all the F_σ are isomorphisms, F is called a *local isomorphism.* If in addition f is a simplicial isomorphism, F is called an *isomorphism.*

Definition I.5.5 (morphism from a complex of groups to a group)**.** Let $G(\mathcal{Y})$ be a complex of groups over a scwol $\mathcal{Y}$ and G a group. A *morphism* $F = (F_\sigma, F(a))$ *from* $G(\mathcal{Y})$ *to* G consists of a homomorphism $F_\sigma : G_\sigma \to G$ for every $\sigma \in V(\mathcal{Y})$ and an element $F(a) \in G$ for each $a \in E(\mathcal{Y})$ such that

- for every $a \in E(\mathcal{Y})$, we have $F_{t(a)}\psi_a = \mathrm{Ad}(F(a))F_{i(a)}$,
- for every pair (a, b) of composable edges of $\mathcal{Y}$, we have $F_{t(a)}(g_{a,b})F(ab) = F(a)F(b)$.

I.5.2 Developability.

Definition I.5.6 (Complex of groups associated to an action without inversion of a group on a simplicial complex [9])**.** Let G be a group acting without inversion by simplicial isomorphisms on a simplicial complex X, let Y be the quotient space and $p : X \to Y$ the natural projection. Up to a barycentric subdivision, we can assume that p restricts to a embedding on every simplex, yielding a simplicial structure on Y. Let $\mathcal{Y}$ be the simplicial scwol associated to Y.

For each vertex σ of $\mathcal{Y}$, choose a simplex $\widetilde{\sigma}$ of X such that $p(\widetilde{\sigma}) = \sigma$. As G acts without inversion on X, the restriction of p to any simplex of X is a homeomorphism on its image. Thus, to every simplex σ' of Y contained in σ, there is a unique τ of X and contained in $\widetilde{\sigma}$, such that $p(\tau) = \sigma'$ To the edge $a = (\sigma, \sigma')$ of $\mathcal{Y}$ we then choose an element $h_a \in G$ such that $h_a.\tau = \widetilde{\sigma'}$. A *complex of groups* $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ *over* Y *associated to the action of* G *on* X is given by the following:

- for each vertex σ of $\mathcal{Y}$, let G_σ be the stabiliser of $\widetilde{\sigma}$,
- for every edge a of $\mathcal{Y}$, the homomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$ is defined by
$$\psi_a(g) = h_a g h_a^{-1},$$

- for every pair (a, b) of composable edges of $\mathcal{Y}$, define

$$g_{a,b} = h_a h_b h_{ab}^{-1}.$$

Moreover, there is an associated morphism $F = (F_\sigma, F(a))$ from $G(\mathcal{Y})$ to G, where $F_\sigma : G_\sigma \to G$ is the natural inclusion and $F(a) = h_a$.

Definition I.5.7 (Developable complex of groups)**.** A complex of groups over a simplicial complex Y is *developable* if it is isomorphic to the complex of groups associated to an action without inversion on a simplicial complex.

Unlike in Bass-Serre theory, not every complex of groups is developable. Checking whether or not a complex of groups is developable is a non trivial problem in general. We will see an algebraic condition that ensures developability in the form of I.5.12, as well as a geometric condition in the form of I.5.18.

I.5.3 The fundamental group of a complex of groups.

We present here two ways to define the fundamental group of a complex of groups. The first one is a generalisation of the notion of an orbifold fundamental group.

Definition I.5.8 ($G(\mathcal{Y})$-loops)**.** A $G(\mathcal{Y})$-*loop* based at σ_0 is a sequence $c = (g_0, e_1, \ldots, e_n, g_n)$ where $(e_1, \ldots, e_n)$ is an edge-path in $\mathcal{Y}$ based at σ_0, and such that $g_0 \in G_{\sigma_0}$ and $g_i \in G_{t(e_i)}$ for $i = 1, \ldots, n$.

If $c' = (g'_0, e'_1, \ldots, e'_m, g'_m)$ is another $G(\mathcal{Y})$-loop, we define the *concatenation* of c and c' as $c * c' = (g_0, e_1, \ldots, e_n, g_n g'_0, e'_1, \ldots, e'_m, g'_m)$.

Definition I.5.9 (homotopy of $G(\mathcal{Y})$-loops)**.** Let $E^+(\mathcal{Y}) = E(\mathcal{Y})$, $E^-(\mathcal{Y})$ be obtained from $E^+(\mathcal{Y})$ by reversing the orientations of the edges of the barycentric subdivision of Y, and set $E^\pm(\mathcal{Y}) = E^+(\mathcal{Y}) \coprod E^-(\mathcal{Y})$. We define the group $FG(\mathcal{Y})$ by the following presentation. It is generated by

$$\coprod_{\sigma \in V(\mathcal{Y})} G_\sigma \coprod E^\pm(\mathcal{Y})$$

subject to the following relations:

- the relations in the groups G_σ,
- $(a^+)^{-1} = a^-$ and $(a^-)^{-1} = a^+$,
- $b^+ a^+ g_{a,b} = (ab)^+$ for a pair of composable edges,
- $\psi_a(g) = a^- g a^+$ for an element $g \in G_{i(a)}$.

We say that two loops are *homotopic* if they have the same image in $FG(\mathcal{Y})$.

Definition I.5.10 (fundamental group of a complex of groups)**.** The *fundamental group* of the complex of groups $G(\mathcal{Y})$ at σ_0, denoted $\pi_1(G(\mathcal{Y}), \sigma_0)$, is the group of homotopy classes of $G(\mathcal{Y})$-loops with the group law induced by the concatenation of $G(\mathcal{Y})$-loops.

There is an alternative way to define the fundamental group of a complex of groups. This version generalises the analogous notion introduced by Serre for graphs of groups [43] and provides an explicit presentation for the group. This is summarised in the following proposition:

Proposition I.5.11 (presentation of the fundamental group of a complex of groups, Proposition 3.2 of [26])**.** *Let $G(\mathcal{Y})$ be a complex of groups over a simplicial complex Y, and let $\mathcal{Y}$ be the associated simplicial scwol. Consider a vertex σ_0 of $\mathcal{Y}$ and a maximal tree T in the 1-skeleton of the first barycentric subdivision of Y. We identify T with the corresponding set of edges of $E(\mathcal{Y})$.*

The fundamental group $\pi_1\big(G(\mathcal{Y}), \sigma_0\big)$ of $G(\mathcal{Y})$ at σ_0 is isomorphic to the abstract group $\pi_1\big(G(\mathcal{Y}), T\big)$ generated by the set

$$\coprod_{\sigma \in V(\mathcal{Y})} G_\sigma \coprod E^{\pm}(\mathcal{Y})$$

and subject to the following relations:

- *the relations in the groups G_σ,*
- $(a^+)^{-1} = a^-$ *and* $(a^-)^{-1} = a^+$,
- $b^+ a^+ g_{a,b} = (ab)^+$ *for a pair of composable edges,*
- $\psi_a(g) = a^- g a^+$ *for an element* $g \in G_{i(a)}$,
- $a^+ = 1$ *for every edge a of T.*

The following proposition provides an algebraic criterion to prove the developability of a complex of groups.

Proposition I.5.12 (Proposition III.$C(\mathcal{Y})$.3.9 of [9])**.** *Let Y be a simplicial complex, $\mathcal{Y}$ its associated simplicial scwol and let $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ be a complex of groups over Y. Let T be a maximal tree in the 1-skeleton of the first barycentric subdivision of Y.*

Then $G(\mathcal{Y})$ is developable if and only if each of the natural homomorphisms $G_\sigma \to \pi_1(G(\mathcal{Y}), T)$ is injective.

I.5.4 The universal covering space of a complex of groups.

Given a developable complex of groups, that is, a complex of groups isomorphic to the one induced by the action of a group G on a simply-connected simplicial complex space X, we present a procedure to recover the simplicial complex acted upon.

Definition I.5.13 (The basic construction)**.** Let $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ be a developable complex of groups over a simplicial complex Y, G a group, and $F = (F_\sigma, F(a))$ a morphism from $G(\mathcal{Y})$ to G.

We define a scwol $D(\mathcal{Y}, F)$ in the following way:

- the vertex set is
$$V(D(\mathcal{Y}, F)) = \big(G \times \coprod_{\sigma \in V(\mathcal{Y})} \{\sigma\}\big)/ \simeq$$
where
$$(gF(g'), \{\sigma\}) \simeq (g, \{\sigma\}) \text{ if } g' \in G_\sigma, g \in G.$$
- the edge set is
$$E(D(\mathcal{Y}, F)) = \big(G \times \coprod_{a \in E(\mathcal{Y})} \{a\}\big)/ \simeq$$
where
$$(gF(g'), \{a\}) \simeq (g, \{a\}) \text{ if } g' \in G_a, g \in G.$$
- the maps $i, t : E(D(\mathcal{Y}, F)) \to V(D(\mathcal{Y}, F))$ are given by
$$i\big([g, a]\big) = \big([g, i(a)]\big),$$
$$t\big([g, a]\big) = \big([gF(a)^{-1}, t(a)]\big),$$
- the composition of edges of $D(\mathcal{Y}, F)$ is given by
$$[g, a][h, b] = [h, ab],$$
where (a, b) is a pair of composable edges of $\mathcal{Y}$ and $g, h \in G$ are such that $g = hF(b)^{-1}$ modulo $F(G_{i(a)})$.

The vertex set $V(D(\mathcal{Y}, F))$ naturally inherits a partially ordered set (briefly a poset) structure as follows: given two vertices σ, σ' of $D(\mathcal{Y}, F)$, we set
$$\sigma \preceq \sigma'$$
if there exist edges $a_1, \dots, a_n$ of $D(\mathcal{Y}, F)$ such that $(a_1, a_2), \dots, (a_{n-1}, a_n)$ are pairs of composable edges, $i(a_1) = \sigma$ and $t(a_n) = \sigma'$.

Finally, we define a simplicial complex $X(\mathcal{Y}, F)$ in the following way:

- vertices of $X(\mathcal{Y}, F)$ are elements of $V(D(\mathcal{Y}, F))$ that are minimal for the partial order $\preceq$,
- vertices $\sigma_0, \ldots, \sigma_k$ of $X(\mathcal{Y}, F)$ span a k-simplex if there exists a vertex τ of $D(\mathcal{Y}, F)$ such that $\sigma_0 \preceq \tau, \ldots, \sigma_k \preceq \tau$.

There is a natural action of G on $V(D(\mathcal{Y}, F))$ given by

$$g.[h, \sigma] = [gh, \sigma].$$

This action preserves the partial order $\preceq$, yielding an action of G on $X(\mathcal{Y}, F)$.

If Y is endowed with a M_κ-complex structure, $\kappa \leq 0$, there is a natural M_κ-complex structure on $X(\mathcal{Y}, F)$.

Note that $X(\mathcal{Y}, F)$ is naturally simplicially isomorphic to the quotient space

$$(G \times \coprod_{\sigma \in V(\mathcal{Y})} \sigma)/ \simeq$$

where

$$(gF(g'), x) \simeq (g, x) \text{ if } x \in \sigma, g' \in G_\sigma,$$
$$(g, i_{\sigma,\sigma'}(x)) \simeq (gF((\sigma, \sigma'))^{-1}, x) \text{ if } x \in \sigma, (\sigma, \sigma') \in E(\mathcal{Y}),$$

and $i_{\sigma,\sigma'} : \sigma \hookrightarrow \sigma'$ is the natural inclusion.

Theorem I.5.14 (Universal covering space of a complex of groups, III.$C(\mathcal{Y})$.3.13, III.$C(\mathcal{Y})$.3.15 of [9])**.** *Let $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ be a developable complex of groups over a simplicial complex Y.*

(*i*) *Let T be a maximal tree in the 1-skeleton of the first barycentric subdivision of Y, and let ι_T be the morphism from $G(\mathcal{Y})$ to $\pi_1(G(\mathcal{Y}), T)$ obtained by mapping each element of the local groups G_σ to the corresponding generator of $\pi_1(G(\mathcal{Y}), T)$ and each edge a of $\mathcal{Y}$ to the corresponding generator of $\pi_1(G(\mathcal{Y}), T)$.*

Then $X(\mathcal{Y}, \iota_T)$ is connected and simply connected. Furthermore, $G(\mathcal{Y})$ is the complex of groups associated to the action of $\pi_1(G(\mathcal{Y}), T)$ on $X(\mathcal{Y}, \iota_T)$, and the morphism from $G(\mathcal{Y})$ to $\pi_1(G(\mathcal{Y}), T)$ associated to that action is ι_T.

(*ii*) *Suppose that $G(\mathcal{Y})$ is the complex of groups associated to the action without inversion by simplicial isometries of a group G on a simply connected simplicial space X, with quotient space Y, and that $F : G(\mathcal{Y}) \to G$ is the associated morphism.*

Then there exists a G-equivariant simplicial isometry $X(\mathcal{Y}, \iota_T) \to X$ over the identity of Y. Such a simplicial complex X is called the universal covering of the complex of groups $G(\mathcal{Y})$. □

I.5.5 Non-positively curved complexes of groups.

We now turn to a geometric condition that ensures the developability of a given complex of groups. From now on, we assume that Y is endowed with a M_κ-structure, $\kappa \leq 0$.

Definition I.5.15 (Local complex of groups)**.** Let v be a vertex of Y. We denote by $G(\mathcal{Y}_v)$ the complex of groups over the star of v induced by $G(\mathcal{Y})$ in the obvious way.

We have the following result:

Proposition I.5.16 (Proposition III.$\mathcal{C}$. 4.11 of [9])**.** *For every vertex v of Y, the local complex of groups $G(\mathcal{Y}_v)$ is developable and its fundamental group is isomorphic to G_σ. Denote by X_v its universal covering, called a local development. Then the M_κ-structure on $St(v)$ yields a M_κ-structure with finitely many isometry types of simplices on X_v such that the G_σ-equivariant projection $X_v \to St(v)$ restricts to an isometry on every simplex.* □

Definition I.5.17 (non-positively curved complex of groups)**.** We say that $G(\mathcal{Y})$ is *non-positively curved* if each local development X_v with the simplicial metric coming from the M_κ-structure of Y is a CAT(0) space.

Theorem I.5.18 (Theorem III.$\mathcal{C}$.4.17 of [9])**.** *If the complex of groups $G(\mathcal{Y})$ is non-positively curved then it is developable.* □

In the case of a non-positively curved complex of groups, we have the following useful proposition.

Proposition I.5.19 (Proposition III.$\mathcal{C}$.4.11 of [9])**.** *Assume that the complex of groups $G(\mathcal{Y})$ is non-positively curved, and let X be its universal covering. Let v be a vertex of Y and $\overline{v}$ a vertex of X that projects to v. Then there is a G_σ-equivariant isometry from the local development X_v to the star of $\overline{v}$ in X.* □

Chapter II

Complexes of spaces and classifying spaces.

Graphs of spaces were introduced by Scott and Wall [42] as a powerful topological tool to study graphs of groups. Generalisations to higher dimensional complexes of groups have been studied by Corson [13] and Haefliger [27] (and, in a different setting, by Davis [16]). In this chapter, we introduce a notion of complex of spaces that is suitable to our purpose. In particular, given a developable complex of groups, we explain how one may use the theory of complexes of spaces to construct a model of classifying space for its fundamental group.

II.1 Complexes of spaces and their topology.

In this section, we introduce the notion of a complex of spaces over a simplicial complex X and study its basic properties.

II.1.1 Complexes of spaces.

Definition II.1.1. A *complex of spaces* $C(\mathcal{X})$ *over* a simplicial complex X consists of the following data:

- for every simplex σ of X, a topological space C_σ, called a *fibre*,
- for every pair of simplices $\sigma \subset \sigma'$, a continuous map $\phi_{\sigma',\sigma} : C_{\sigma'} \to C_\sigma$, called a *gluing map*, such that for every $\sigma \subset \sigma' \subset \sigma''$, we have $\phi_{\sigma,\sigma''} = \phi_{\sigma,\sigma'} \circ \phi_{\sigma',\sigma''}$.

If all the fibres are CW-complexes and all the gluing maps are cellular, we will speak of a *complex of CW-complexes*. If all the fibres are pointed spaces and all the maps are pointed, we will speak of a complex of *pointed* spaces.

Definition II.1.2 (realisation of a complex of spaces)**.** Let $C(\mathcal{X})$ be a complex of spaces over a simplicial complex X. The *realisation* of $C(\mathcal{X})$ is the quotient space

$$|C(\mathcal{X})| = \Big(\coprod_{\sigma \in S(X)} \sigma \times C_\sigma \Big) / \simeq$$

where

$$(i_{\sigma,\sigma'}(x), s) \simeq (x, \phi_{\sigma,\sigma'}(s)) \text{ for } x \in \sigma \subset \sigma' \text{ and } s \in C_{\sigma'},$$

and $i_{\sigma,\sigma'} : \sigma \hookrightarrow \sigma'$ is the natural inclusion. The class in $|C(\mathcal{X})|$ of a point (x, s) will be denoted $[x, s]$.

Definition II.1.3. A complex of spaces $C(\mathcal{X})$ over a simplicial complex X will be called *locally finite* if for every simplex σ of X and every point $x \in C_\sigma$, there exists an open set U of C_σ containing x and such that there are only finitely many simplices σ' containing σ and satisfying $U \cap \mathrm{Im}(\phi_{\sigma,\sigma'}) \neq \varnothing$.

Proposition II.1.4. *Let $C(\mathcal{X})$ be a locally finite complex of CW-complexes over a simplicial complex X. Then $|C(\mathcal{X})|$ admits a natural locally finite CW-complex structure, for which the $\sigma \times C_\sigma$ embed as subcomplexes.* □

II.1.2 Topology of complexes of spaces with contractible fibres.

Notation. In this paragraph, we will say that a complex of spaces is *pointed* if each fiber comes with a chosen basepoint. Note that we do not require the maps to preserve the basepoints (hence this does not necessarily yield a complex of pointed spaces).

Definition II.1.5. Let $C(\mathcal{X})$ be a pointed complex of CW-complexes over a simplicial complex X and $Y \subset X$ a subcomplex. We denote by $C_Y(\mathcal{X})$ the pointed complex of CW-complexes over X defined as follows:

- Let $(C_Y)_\sigma = C_\sigma$ if $\sigma \nsubseteq Y$, $(C_Y)_\sigma$ is the basepoint of C_σ otherwise,
- For $\sigma \subset \sigma'$, let $\phi^Y_{\sigma,\sigma'}$ be the composition $(C_Y)_{\sigma'} \to C_{\sigma'} \xrightarrow{\phi_{\sigma,\sigma'}} C_\sigma \twoheadrightarrow (C_Y)_\sigma$.

We denote by $p_Y : |C(\mathcal{X})| \to |C_Y(\mathcal{X})|$ the canonical projection, and simply p for $p_X : |C(\mathcal{X})| \to X$. In the same way, if $Y \subset Y'$ are subcomplexes of X, we denote by $p_{Y,Y'} : |C_Y(\mathcal{X})| \to |C_{Y'}(\mathcal{X})|$ the canonical projection.

Lemma II.1.6. *Let $C(\mathcal{X})$ be a pointed complex of CW-complexes over a simplicial complex X. Let Y be a finite subcomplex of X such that for every simplex σ of Y, the fibre C_σ is contractible. Then $p_Y : |C(\mathcal{X})| \to |C_Y(\mathcal{X})|$ is a homotopy equivalence.*

Proof. It amounts to proving the result for Y consisting of a single closed simplex σ. We have the following commutative diagram:

$$\begin{array}{ccc} |C(\mathcal{X})| & \xrightarrow{p_Y} & |C_Y(\mathcal{X})| \\ \simeq \downarrow & & \downarrow \simeq \\ |C(\mathcal{X})|/(\sigma \times C_\sigma) & \xrightarrow[=]{} & |C_Y(\mathcal{X})|/(\sigma \times \star)\,. \end{array}$$

The vertical arrows are homotopy equivalences, since we are quotienting by contractible sub-CW-complexes, hence the result. □

Recall that by I.2.5, every simplicial complex X of finite dimension can be given a piecewise-Euclidean metric structure, by identifying each n-dimensional simplex of X with the standard n-dimensional simplex of $\mathbb{R}^n$.

Theorem II.1.7 (Dowker [18])**.** *Let X be a simplicial complex of finite dimension. Then the (continuous) identity map $X \to X$ from X with its CW topology to X with its piecewise-Euclidean metric is a homotopy equivalence.* □

Proposition II.1.8. *Let $C(\mathcal{X})$ be a complex of CW-complexes with contractible fibres over a simplicial complex X of finite dimension. Then the projection $p : |C(\mathcal{X})| \to X$ is a homotopy equivalence.*

Proof. Endow X with its canonical piecewise-Euclidean metric, and endow $C(\mathcal{X})$ with a structure of pointed complex of CW-complexes. By the previous theorem, it is enough to show that the projection $p : |C(\mathcal{X})| \to X$ induces isomorphisms on homotopy groups, when X is endowed with its CW topology. For that topology, a continuous map from a compact space to X has its image contained in a finite subcomplex, to which Lemma II.1.6 applies. □

II.2 Constructing classifying spaces out of complexes of spaces.

In this section, given a developable complex of groups $G(\mathcal{Y})$ over a finite simplicial complex Y, we build a classifying space for its fundamental group. In what follows, $G(\mathcal{Y})$ is a non-positively curved complex of groups $G(\mathcal{Y})$ over a finite simplicial complex endowed with a M_κ-structure, $\kappa \leq 0$.

Notation: Recall that a complex of groups consists of the data $(G_\sigma, \psi_a, g_{a,b})$ of local groups (G_σ), local maps (ψ_a) and twisting elements $(g_{a,b})$. From now on, given an inclusion $\sigma \subset \sigma'$ of simplices, we will often write $\psi_{\sigma,\sigma'}$ instead of $\psi_{(\sigma,\sigma')}$. Similarly, given an inclusion $\sigma \subset \sigma' \subset \sigma''$, we will sometimes write $g_{\sigma,\sigma',\sigma''}$ instead of $g_{(\sigma,\sigma'),(\sigma',\sigma'')}$. We fix a maximal tree T in the 1-skeleton of the first barycentric subdivision of Y, which allows us to define the fundamental group $G = \pi_1(G(\mathcal{Y}), T)$ and the canonical morphism $\iota_T : G(\mathcal{Y}) \to G$ given by the collection of injections $G_\sigma \to G$. Finally, we define X as the universal covering of $G(\mathcal{Y})$ associated to ι_T. The simplicial complex X naturally inherits a M_κ-structure from that of Y and the simplicial metric d on X makes it a complete geodesic metric space. This space is CAT(0) by the curvature assumption on $G(\mathcal{Y})$.

Definition II.2.1. A *complex of classifying spaces $EG(\mathcal{Y})$ compatible with the complex of groups* $G(\mathcal{Y})$ consists of the following:

- For every vertex σ of $\mathcal{Y}$, a space EG_σ that is a model of classifying space for proper actions of the local group G_σ,
- For every edge a of $\mathcal{Y}$ with initial vertex $i(a)$ and terminal vertex $t(a)$, a $G_{i(a)}$-equivariant map $\phi_a : EG_{i(a)} \to EG_{t(a)}$, that is, for every $g \in G_{i(a)}$ and every $x \in EG_{i(a)}$, we have
$$\phi_a(g.x) = \psi_a(g).\phi_a(x),$$
and such that for every pair (a, b) of composable edges of $\mathcal{Y}$, we have:
$$g_{a,b} \circ \phi_{ab} = \phi_a \phi_b,$$

We emphasise that a complex of classifying spaces compatible with the complex of groups $G(\mathcal{Y})$ is *not* a complex of spaces over Y if the twisting elements $g_{a,b}$ are not tivial. Nonetheless, this data is used to build a complex of spaces over X, as explained in the following definition.

Definition II.2.2. Suppose that there exists a complex of classifying spaces $EG(\mathcal{Y})$ compatible with $G(\mathcal{Y})$. We define the space
$$Cl_{EG(\mathcal{Y})} = \Big(G \times \coprod_{\sigma \in V(\mathcal{Y})} (\sigma \times EG_\sigma)\Big)/ \simeq$$
where
$$(g, i_{\sigma,\sigma'}(x), s) \simeq \Big(g\iota_T\big((\sigma,\sigma')\big)^{-1}, x, \phi_{(\sigma,\sigma')}(s)\Big) \text{ if } (\sigma,\sigma') \in E(\mathcal{Y}), x \in \sigma', g \in G,$$
$$(gg', x, s) \simeq (g, x, g's) \text{ if } x \in \sigma, s \in EG_\sigma, g' \in G_\sigma, g \in G.$$
The canonical projection $G \times \coprod_{\sigma \in V(\mathcal{Y})} (\sigma \times EG_\sigma) \to G \times \coprod_{\sigma \in V(\mathcal{Y})} \sigma$ yields a map
$$p : Cl_{EG(\mathcal{Y})} \to X.$$

The action of G on $G \times \coprod_{\sigma \in V(\mathcal{Y})} (\sigma \times EG_\sigma)$ on the first factor by left multiplication yields an action of G on $Cl_{EG(\mathcal{Y})}$, making the projection $p : Cl_{EG(\mathcal{Y})} \to X$ a G-equivariant map.

Note that $Cl_{EG(\mathcal{Y})}$ can be seen as the realisation a complex of spaces over X, the fibre of a simplex $[g, \sigma]$ being the classifying space EG_σ. Indeed, for en edge $[g, a]$ of the first barycentric subdivision of X, the gluing map $\phi_{[g\iota_T(a)^{-1}, i(a)], [g, t(a)]} : EG_{i(a)} \to EG_{t(a)}$ is defined as $\phi_{i(a), t(a)}$.

For every simplex σ of X, we denote by EG_σ the fibre over σ of that complex of space. For simplices σ, σ' of X such that $\sigma \subset \sigma'$, we denote by $\phi_{\sigma, \sigma'} : EG_{\sigma'} \to EG_\sigma$ the associated gluing map.

Theorem II.2.3. *If $EG(\mathcal{Y})$ is a complex of classifying spaces compatible with $G(\mathcal{Y})$, then the space $Cl_{EG(\mathcal{Y})}$ is a classifying space for proper actions of G.*

Proof. Local finiteness: Let σ be a simplex of X and U be an open set of EG_σ that is relatively compact. It is enough to prove that for any injective sequence (σ_n) of simplices of X containing σ there are only finitely many n such that the image of ϕ_{σ, σ_n} meets U. By cocompactness of the action, we can assume that all the σ_n are in the same G-orbit, and let σ be a simplex in that orbit. Since the action of G_σ on EG_σ is proper, it follows that for every simplex σ' containing σ and every compact subset K of EG_σ, only finitely many distinct translates $gEG_{\sigma'}$ in EG_σ can meet K, hence the result.

Contractibility: The space $Cl_{EG(\mathcal{Y})}$ has the same homotopy type as X by II.1.8, which is contractible since $G(\mathcal{Y})$ is non-positively curved.

Cocompact action: For every simplex σ of Y, we choose a compact fundamental domain K_σ for the action of G_σ on $D_\sigma = EG_\sigma$. Now the image in $Cl_{EG(\mathcal{Y})}$ of $\bigcup_{\sigma \in S(Y)} \sigma \times K_\sigma$ clearly defines a compact subset of $Cl_{EG(\mathcal{Y})}$ meeting every G-orbit.

Proper action: As $Cl_{EG(\mathcal{Y})}$ is a locally finite CW-complex, hence a locally compact space, it is enough to show that every finite subcomplex intersects only finitely many of its G-translates.
Let us first show that for every cell τ of $Cl_{EG(\mathcal{Y})}$, there are only finitely many $g \in G$ such that $g\tau = \tau$. Indeed, let $g \in G$ such that $g\tau = \tau$. The canonical projection $Cl_{EG(\mathcal{Y})} \to X$ is G-equivariant and sends a cell of $Cl_{EG(\mathcal{Y})}$ on a simplex of X, thus g also stabilises the simplex $p(\tau) \subset X$. Since G acts without inversion on X, g pointwise stabilises the vertices of $p(\tau)$. Let s be such a vertex. Then $g \in G_s$ and, by construction of $Cl_{EG(\mathcal{Y})}$, the action of G_s on $Cl_{EG(\mathcal{Y})}$ induces on EG_s the natural action of G_s on EG_s. Thus, by definition of a classifying space for proper actions, this implies that there are only finitely many possibilities for g.

Now, let F be a finite subcomplex of $Cl_{EG(\mathcal{Y})}$ and $S(F)$ the (finite) set of pairs (τ, τ') of cells of F that are in the same G-orbit. The set $\{g \in G \mid gF \cap F \neq \varnothing\}$ is contained in $\bigcup_{(\tau, \tau') \in S(F)} \{g \in G \mid g\tau = \tau'\}$, and $\{g \in G \mid g\tau = \tau'\}$ has the same cardinality as the set $\{g \in G \mid g\tau = \tau\}$, which is finite by the previous argument.

Fixed sets: Let H be a finite subgroup of G. As G acts without inversion on the CAT(0) complex X, the subset X^H is a nonempty convex subcomplex of X. Furthermore, for every simplex σ of X^H, the subcomplex $(EG_\sigma)^H$ of EG_σ is nonempty and contractible. Thus $Cl^H_{EG(\mathcal{Y})}$ is the realisation of a complex of spaces over the contractible complex X^H and with contractible fibres, hence it is nonempty and contractible by II.1.8.

If H is an infinite subgroup of G, we prove by contradiction that $Cl^H_{EG(\mathcal{Y})}$ is empty. If this was not the case, there would exist a simplex σ fixed pointwise under H and a point x of EG_σ that is fixed under $H \subset G_\sigma$. But this is absurd as $(EG_\sigma)^H = \varnothing$ by assumption. $\square$

Remark: Theorem II.2.3 still holds if, instead of assuming that the complex of groups $G(\mathcal{Y})$ is non-positively curved, we simply assume $G(\mathcal{Y})$ to be developable and such that the fixed set of any finite subgroup of its fundamental group is contractible.

Chapter III

Metric small cancellation over graphs of groups.

In this chapter, we study metric small cancellation groups over a finite graph of groups. This yields a new class of groups acting on 2-dimensional CAT(0) complexes and which admit a cocompact model of classifying space for proper actions.

Theorem III.0.4. *Let $G(\Gamma)$ be a finite graph of groups over a finite simplicial graph Γ, with fundamental group G and Bass-Serre tree T, and such that no non-trivial element of G fixes a line of T. Let $\mathcal{R}$ be a finite symmetrized collection of hyperbolic elements of G satisfying the small cancellation condition $C''(\frac{1}{6})$ for the action of G on T. Then the quotient ${}^{G}/_{\ll \mathcal{R} \gg}$ is the fundamental group of a non-positively curved 2-dimensional complex of groups whose local groups are either finite or subgroups of the local groups of $G(\Gamma)$.*

Theorem III.0.5. *Let $G(\Gamma)$ be a graph of groups satisfying the hypotheses of III.0.4. If there exists a graph of classifying spaces compatible with $G(\Gamma)$, then ${}^{G}/_{\ll \mathcal{R} \gg}$ admits a cocompact model of classifying space for proper actions.*

Here is an outline of the chapter. The first section contains gluing constructions for complexes of groups which are reminiscent of the theory of orbispaces introduced by Haefliger [26]. Section 3.2 is an introduction to small cancellation theory, and presents the theory of small cancellation over a graph of groups from a geometric viewpoint. Given a small cancellation group ${}^{G}/_{\ll \mathcal{R} \gg}$ over a finite graph of groups, we construct in Section 3.3 various examples of developable 2-complexes of groups that admit ${}^{G}/_{\ll \mathcal{R} \gg}$ as their fundamental group. Using the theory of complexes of spaces developed in the previous chapter, this is used to construct a classifying space for proper actions for ${}^{G}/_{\ll \mathcal{R} \gg}$.

III.1 Trees of complexes of groups.

In this section, we explain how, given a simplicial complex Y, subcomplexes (Y_i) whose interiors cover Y and such that the nerve of the associated open cover is a tree, and a family of complexes of groups $G(\mathcal{Y}_i)$ over Y_i, one can glue them together to obtain a complex of groups $G(\mathcal{Y})$ over Y. This procedure can be thought as making "trees of complexes of groups". In order to lighten notations, we will only detail the case of a cover consisting of two subcomplexes with a connected intersection.

III.1.1 Immersions of complexes of groups.

Definition III.1.1 (Immersion of complexes of groups.)**.** Let $G(\mathcal{Y}_1)$ and $G(\mathcal{Y}_2)$ be two complexes of groups over two simplicial complexes Y_1 and Y_2 and $F : G(\mathcal{Y}_1) \to G(\mathcal{Y}_2)$ a morphism of complexes of groups over a simplicial morphism $f : Y_1 \to Y_2$. We say that F is an *immersion* if f is a simplicial immersion and all the local maps F_σ are injective.

Note that if in addition both complexes of groups are assumed to be developable, then the simplicial immersion f lifts to an equivariant simplicial immersion between their universal coverings.

For $i = 1, 2$, let X_i, Y_i be simplicial complexes, G_i a group acting without inversion on X_i by simplicial isomorphisms, $p_i : X_i \to Y_i$ a simplicial map factoring through X_i/G_i and inducing a simplicial isomomorphism $X_i/G_i \simeq Y_i$. Suppose we are given a simplicial immersion $f : Y_1 \to Y_2$, a homomorphism $\alpha : G_1 \to G_2$ and an equivariant simplicial immersion $\overline{f} : X_1 \to X_2$ over f such that for every simplex σ of X_1, the induced map $\alpha : \mathrm{Stab}(\sigma) \to \mathrm{Stab}(\bar{f}(\sigma))$ is a monomorphism. Recall that from the action of G_i on X_i, it is possible to define a complex of groups over Y_i that encodes it. We now explain how to define such complexes of groups $G(\mathcal{Y}_1)$ and $G(\mathcal{Y}_2)$ over Y_1 and Y_2, such that there is an immersion $G(\mathcal{Y}_1) \to G(\mathcal{Y}_2)$.

Recall that to define a complex of groups over Y_1 induced by the action of G_1 on X_1, we had to associate to every vertex σ of $\mathcal{Y}_1$ a simplex $\overline{\sigma}$ of X_1, and to every edge a of $\mathcal{Y}_1$ an element h_a of G_1 (see I.5.6). Assume we have made such choices to define $G(\mathcal{Y}_1)$. We now make such choices for Y_2.

- Let σ' be a vertex of $\mathcal{Y}_2$, which we identify with the associated simplex of Y_2. If $\sigma' = f(\sigma)$ for a simplex σ of Y_1, we choose $\overline{\sigma'} = \overline{f}(\overline{\sigma})$. Otherwise, we pick an arbitrary lift of σ'.

- Let a' be an edge of $\mathcal{Y}_2$. If $a' = f(a)$ for an edge a of $\mathcal{Y}_1$, we choose $h_{a'} = \alpha(h_a)$. Otherwise, we choose an arbitrary element $h_{a'}$ that satisfyies the conditions described in I.5.6.

This yields a complex of groups $G(\mathcal{Y}_2)$ over Y_2. We now define a morphism of complex of groups $F = (F_\sigma, F(a)) : G(\mathcal{Y}_1) \to G(\mathcal{Y}_2)$ over f as follows:

- The maps $F_\sigma : G_\sigma \to G_{f(\sigma)}$ are the monomorphisms $\alpha : \mathrm{Stab}(\overline{\sigma}) \to \mathrm{Stab}(\overline{f}(\overline{\sigma}))$,
- The elements $F(a)$ are all trivial.

It is straightforward to check that this indeed yields an immersion $F = (F_\sigma, F(a)) : G(\mathcal{Y}_1) \to G(\mathcal{Y}_2)$ over f.

Definition III.1.2. We call the immersion $F = (F_\sigma, F(a)) : G(\mathcal{Y}_1) \to G(\mathcal{Y}_2)$ over f an immersion *associated to* $(Y_1 \overset{f}{\to} Y_2, X_1 \overset{\bar{f}}{\to} X_2, G_1 \overset{\alpha}{\to} G_2)$.

III.1.2 Amalgamation of non-positively curved complexes of groups.

In what follows, Y is a finite simplicial complex, Y_1, Y_2 are subcomplexes of Y whose interiors cover Y, and $Y_0 = Y_1 \cap Y_2$. We assume that Y_0 is connected. We further assume that, for $i = 0, 1, 2$, we are given a simplicial complex X_i, a G_i a group acting without inversion on X_i, $p_i : X_i \to Y_i$ a simplicial map factoring through X_i/G_i and inducing a simplicial isomorphism $X_i/G_i \simeq Y_i$. We assume that, for $i = 1, 2$, we are given a homomorphism $\alpha_i : G_0 \to G_i$ and an equivariant simplicial immersion $\bar{f}_i : X_0 \to X_i$ over the inclusion $f_i : Y_0 \hookrightarrow Y_i$ such that for every simplex σ of X_0, the induced map $\alpha_i : \mathrm{Stab}(\sigma) \to \mathrm{Stab}(\bar{f}_i(\sigma))$ is an isomorphism.

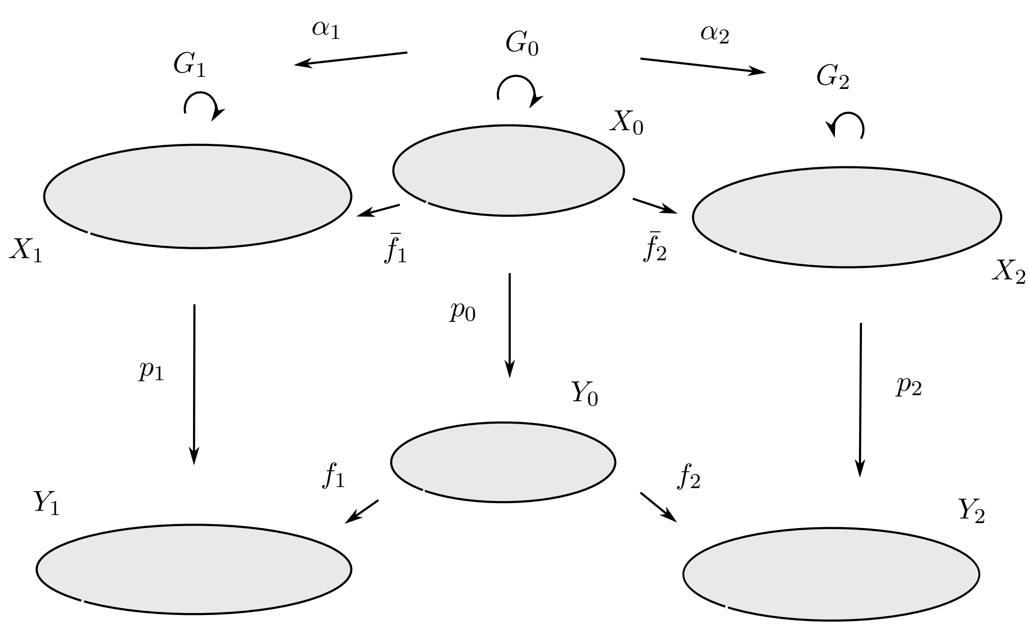

Figure III.1 - A diagram of maps.

By the results of the previous paragraph, we can choose complexes of groups $G(\mathcal{Y}_i)$ over Y_i associated to these actions in such a way that there are immersions $G(\mathcal{Y}_0) \overset{F_i}{\to} G(\mathcal{Y}_i)$ associated to $(Y_0 \overset{f_i}{\to} Y_i, X_0 \overset{\bar{f}_i}{\to} X_i, G_0 \overset{\alpha_i}{\to} G_i)$. Note that the local maps $(F_i)_\sigma$ are isomorphisms.

We now use these immersions to amalgamate $G(\mathcal{Y}_1)$ and $G(\mathcal{Y}_2)$ along $G(\mathcal{Y}_0)$. So as to emphasize which complex of groups is under consideration, we will indicate it as a superscript (see below). We define a complex of groups $G(\mathcal{Y})$ over Y as follows:

- If σ is a vertex of $\mathcal{Y}_0$, we set $G_\sigma^Y = G_\sigma^{Y_0}$.

  If σ is a vertex of $\mathcal{Y}_i \setminus \mathcal{Y}_0$, we set $G_\sigma^Y = G_\sigma^{Y_i}$.
- If a is an edge of $\mathcal{Y}_0$, we set $\psi_a^Y = \psi_a^{Y_0}$.

  If a is an edge of $\mathcal{Y}_i \setminus \mathcal{Y}_0$, we set $\psi_a^Y = \psi_a^{Y_i}$.

  If a is an edge of $\mathcal{Y}_i$ such that $i(a)$ is a vertex of $\mathcal{Y}_i \setminus \mathcal{Y}_0$ and $t(a)$ is a vertex of $\mathcal{Y}_0$, we set $\psi_a^Y = \big((F_i)_{t(a)}\big)^{-1} \circ \psi_a^{Y_i}$.
- If (a, b) is a pair of composable edges of $\mathcal{Y}_0$, we set $g_{a,b}^Y = g_{a,b}^{Y_0}$.

  If (a, b) is a pair of composable edges of $\mathcal{Y}_i \setminus \mathcal{Y}_0$, we set $g_{a,b}^Y = g_{a,b}^{Y_i}$.

  If (a, b) is a pair of composable edges of $\mathcal{Y}$ such that b is not an edge of $\mathcal{Y}_0$ but $t(a)$ is a vertex of $\mathcal{Y}_0$, we set $g_{a,b}^Y = \big((F_i)_{t(a)}\big)^{-1}(g_{a,b}^{Y_i})$.

Definition III.1.3 (Amalgamation of complexes of groups)**.** We denote by $G(\mathcal{Y}_1) \underset{G(\mathcal{Y}_0)}{\cup} G(\mathcal{Y}_2)$ the previous complex of groups.

Theorem III.1.4 (Seifert-van Kampen Theorem for complexes of groups, Theorem III.$\mathcal{C}$.3.11.(5) of [9])**.** *With the same notations as above, the fundamental group of* $G(\mathcal{Y}_1) \underset{G(\mathcal{Y}_0)}{\cup} G(\mathcal{Y}_2)$ *is isomorphic to the pushout* $G_1 \underset{G_0}{*} G_2$. □

We now assume in addition that Y comes equipped with an M_κ-simplicial structure ($\kappa \leq 0$). This endows X_0, X_1, X_2 with an M_κ-simplicial structure that turns the maps $\overline{f_i} : X_0 \to X_i$ into local isometries. Let v be a vertex of Y. Since the interiors of Y_1 and Y_2 cover Y, the star of v is fully contained in one of these subcomplexes. We thus obtain from I.5.18 and I.5.19 the following developability theorem:

Theorem III.1.5. *Under the same assumptions as above, if* X_1 *and* X_2 *are CAT(0) for their induced* M_κ*-structure, then* $G(\mathcal{Y}_1) \underset{G(\mathcal{Y}_0)}{\cup} G(\mathcal{Y}_2)$ *is non-positively curved, hence developable.* □

As stated at the beginning of this section, this theorem generalises directly to the case of a finite complex Y covered by the interiors of a finite family of subcomplexes (Y_i) such that the nerve of the associate open cover is a tree. We give the following particular case, which will be used in the article.

Let Y be a finite simplicial complex endowed with a M_κ-structure ($\kappa \leq 0$), let $Y_0, Y_1, \ldots, Y_n$ be a family of connected subcomplexes of Y whose interiors cover Y, and for $i = 1, \ldots, n$, let $Y_i' = Y_0 \cap Y_i$. We assume that each Y_i' is non empty and connected. We further assume that for $1 \leq i \neq j \leq n$, we have $Y_i \cap Y_j = \varnothing$.

For each Y_i (resp. Y_i'), we are given a simplicial complex X_i (resp. X_i'), a group G_i (resp. G_i') acting without inversion on X_i (resp. X_i'), $p_i : X_i \to Y_i$ (resp. $p_i' : X_i' \to Y_i'$) a simplicial map factoring through X_i/G_i) (resp. X_i'/G_i') and inducing a simplicial isomorphism $X_i/G_i \simeq Y_i$ (resp. $X_i'/G_i' \simeq Y_i'$). This yields an M_κ-structure on each X_i. We assume that, for $i = 1, \ldots, n$, we are given homomorphisms $\alpha_i : G_i' \to G_0$ and $\beta_i : G_i' \to G_i$, an α_i-equivariant simplicial immersion $\overline{f_i} : X_i' \to X_0$ over the inclusion $Y_i' \hookrightarrow Y_0$ and a β_i-equivariant simplicial immersion $\overline{g_i} : X_i' \to X_i$ over the inclusion $Y_i' \hookrightarrow Y_i$. We finally assume that for every simplex σ of X_i', the induced maps $\alpha_i : \mathrm{Stab}(\sigma) \to \mathrm{Stab}(\bar{f}_i(\sigma))$ and $\beta_i : \mathrm{Stab}(\sigma) \to \mathrm{Stab}(\bar{g}_i(\sigma))$ are isomorphisms.

As before, we can construct induced complexes of groups $G(\mathcal{Y}_i)$ over Y_i and $G(\mathcal{Y}_i')$ over Y_i', along with immersions $G(\mathcal{Y}_i') \to G(\mathcal{Y}_0)$ and $G(\mathcal{Y}_i') \to G(\mathcal{Y}_i)$. These complexes of groups can in turn be amalgamated to obtain a complex of groups $G(\mathcal{Y})$ over Y. We get the following:

Theorem III.1.6. *If each simplicial complex X_i, $i = 0, \ldots, n$, is CAT(0) for its induced M_κ-structure, then $G(\mathcal{Y})$ is non-positively curved, hence developable.* □

III.2 Actions on trees and metric small cancellation theory.

III.2.1 Ordinary small cancellation theory.

We present metric small cancellation theory from a geometric viewpoint. For a more combinatorial approach, we refer to [34]. Let F_n be te free group on n generators, acting freely cocompactly on the associated $2n$-valent tree T_n. To every element g of F_n corresponds an isometry of T_n. When g is non-trivial, such an isometry is *hyperbolic*, that is, it admits an invariant embedded line, called the *axis* of g and denoted $A(g)$, on which it acts by translation. The associated translation length, denoted $l(g)$, is the minimal number of generators necessary to obtain an element in the conjugacy class of g.

Let $\mathcal{R}$ be a finite set words of F_n. We assume that $\mathcal{R}$ is *symmetrized*, that is, inverses and cyclic conjugates of elements of $\mathcal{R}$ belong to $\mathcal{R}$.

Definition III.2.1 (Small Cancellation Condition)**.** We say that a symmetrized set $\mathcal{R}$ of elements satisfies the *small cancellation condition* $C'(\lambda)$, with $\lambda > 0$, if for every $r, r' \in \mathcal{R}$

and $g \in F_n$ such that $A(r) \neq g.A(r')$, the diameter of $A(r) \cap g.A(r')$ is strictly less than $\lambda \cdot \min\big(l(g), l(g')\big)$.

The following result is a fundamental theorem of small cancellation theory. Rips remarked that small cancellation groups have δ-thin geodesic triangles [39]. Hyperbolic groups were later introduced by Gromov [23] so as to treat simultaneously a rich class of groups, ranging from small cancellation groups to fundamental groups of compact manifolds with negative curvature.

Theorem III.2.2. *Let $\mathcal{R}$ a finite symmetrized set of elements satisfying the $C'(1/6)$ condition. Then $G = {}^{F_n}/_{\ll \mathcal{R} \gg}$ is a hyperbolic group. Moreover, its presentation Cayley complex is aspherical.* □

III.2.2 Actions on trees and small cancellation theory over a graph of groups.

This geometric approach to small cancellation theory extends to more general group actions on trees. Let G be a group acting without inversion by simplicial isometries on a simplicial tree T. By the classification theorem for isometries of a tree (see [12]) an element g of G either fixes a vertex of T or is *hyperbolic*, that is, it admits an invariant embedded line, called the *axis* of g and denoted $A(g)$, on which it acts by translation. In the latter case, the associated translation length, denoted $l(g)$, is called the *translation length* of g, or simply the *length* of g.

Let $\mathcal{R}$ be a finite set of hyperbolic elements of G. We denote by $R_{\min}$ the minimal length of an element of $\mathcal{R}$.

Definition III.2.3 (Metric Small Cancellation Condition)**.** We say that a finite set $\mathcal{R}$ of hyperbolic elements satisfies the *metric small cancellation condition* $C''(\lambda)$, with $\lambda > 0$, if:

- for every two distinct elements $r, r' \in \mathcal{R}$ and every $g \in G$, the diameter of $A(r) \cap g.A(r')$ is strictly less than $\lambda.R_{\min}$,
- for every $r \in \mathcal{R}$ and $g \in G \setminus \mathrm{Stab}\big(A(r)\big)$, the diameter of $A(r) \cap g.A(r)$ is strictly less than $\lambda.R_{\min}$.

We denote by $l_{\max}$ the maximum of the diameter of these intersections.

This small cancellation condition is much stronger than the usual metric condition $C'(\lambda)$ that is used in ordinary small cancellation theory, but coincides with the usual notion when all the elements of $\mathcal{R}$ have the same translation length. In particular, $\mathcal{R}$ is necessarily finite in the context of metric small cancellation, whereas there are examples of non finitely presented groups in ordinary small cancellation theory.

III.2.3 Some preliminaries on Bass-Serre theory.

We recall here a few facts about Bass-Serre theory. The conventions in this setting (see [43]) are slightly different from the ones described in Chapter I to study complexes of groups.

Let Γ be a simplicial graph and v_0 a chosen vertex. Let $G(\Gamma) = (G_\sigma, \psi_a)$ be a graph of groups over Γ. To the simplicial graph Γ we associate a scwol $\mathcal{D}\Gamma$ obtained by "doubling" every edge as follows:

- The vertex set of $\mathcal{D}\Gamma$ is the set of vertices of Γ.
- The edge set of $\mathcal{D}\Gamma$ is the set of oriented edges of Γ, that is, pairs of the form (v, e) where e in an edge of Γ containing the vertex v. The initial vertex of an edge (v, e) of $\mathcal{D}\Gamma$ is defined as the vertex of e other than v, the terminal vertex as v. For an edge $e = [v, w]$ of Γ, we write $(v, e) = (w, e)^{-1}$.

The fundamental group $\pi_1(G(\mathcal{D}\Gamma), v_0) =: G$ is the set of homotopy classes of $G(\mathcal{D}\Gamma)$-loops, where two $G(\mathcal{D}\Gamma)$-loops are homotopic if they have the same image in the abstract group generated by

$$\coprod_{v\in V(\mathcal{D}\Gamma)} G_v \coprod E(\mathcal{D}\Gamma)$$

and subject to the following relations:

- the relations in the groups G_v,
- $aa^{-1} = a^{-1}a = 1$ for every edge $a \in E(\mathcal{D}\Gamma)$,
- $\psi_{(v,e)}(g) = a^{-1}ga$ for an edge $a = (v, e)$ of $\mathcal{D}\Gamma$ and an element $g \in G_e$.

The group G is isomorphic to $\pi_1(G(\Gamma), v_0)$.

Let τ be a maximal tree in the first barycentric subdivision of Γ. Let T be the associated Bass-Serre tree, that is, the universal covering space of $G(\Gamma)$ with respect to T. Choose a lift $\widetilde{\tau}$ of τ to T and let T_0 be the minimal subtree of T containing $\widetilde{\tau}$. This yields a fundamental domain for the action of G on T. Let $\mathcal{T}(\Gamma)$ be the set of $G(\mathcal{D}\Gamma)$-paths of the form $g_0e_1g_1\ldots e_n$. This set comes with a $\pi_1(G(\mathcal{D}\Gamma), v_0)$-action on the left. To an element $g_0e_1g_1\ldots e_n$ of $\mathcal{T}(\Gamma)$, we associate an edge of T as follows: Let γ be an edge-path from v_0 to $t(e_n)$ which is contained in τ, and γ^{-1} the reverse edge-path. The $G(\mathcal{D}\Gamma)$-loop $g_0e_1g_1\ldots e_n\gamma^{-1}$ defines an element of G, and we associate to $g_0e_1g_1\ldots e_n$ the edge $(g_0e_1g_1\ldots e_n\gamma^{-1})\cdot\widetilde{e_n}$ of T, where $\widetilde{e_n}$ is the unique (oriented) lift of e_n contained in T_0.

In what follows, we will sometimes speak of the edge $g_0e_1g_1\ldots e_n$, so as to avoid writing $(g_0e_1g_1\ldots e_n\gamma^{-1})\cdot\widetilde{e_n}$.

Lemma III.2.4. *The aforementioned map $\mathcal{T}(\Gamma)\to T$ is G-equivariant.* □

III.3 Complexes of groups arising from small cancellation theory.

From now on, we consider a graph of groups $G(\Gamma)$ over a finite graph Γ, with fundamental group G and associated Bass-Serre tree T, which satisfies the assumptions of Theorem III.0.4. We consider a finite symmetrized set of hyperbolic elements $\mathcal{R}$ satisfying the metric small cancellation condition $C''(\frac{1}{6})$. With the above notation, this simply reads $l_{\max} < \frac{1}{6}R_{\min}$.

III.3.1 A non-positively curved complex of groups.

The cone-off construction. Let $\Lambda = \{g.A(w), w \in \mathcal{R}\}$ be the collection of translates of axes of elements of $\mathcal{R}$, seen as a collection of subsets of the Bass-Serre tree T. Note that Λ is G-invariant. For every $\lambda \in \Lambda$, let C_λ be the cone over λ, with a simplicial structure coming from that of λ and in which every triangle is modelled after a flat isosceles triangle τ whose basis has length 1 and whose other edges have length $r = \left(2\sin(\frac{\pi}{R_{\min}})\right)^{-1}$ (the angle at the apex O_λ being $\frac{2\pi}{R_{\min}}$). Let $\widehat{T}$ be the 2-dimensional simplicial complex obtained from the disjoint union of T and the C_λ's by identifying the base of C_λ with $\lambda \subset T$ in the obvious way. The action of G on T naturally extends to $\widehat{T}$.

Coordinates. We introduce some coordinates as follows. For an element $u \in C_\lambda \subset \widehat{T}$, we write $u = [\lambda, x, t]$, where x is the intersection point of the ray $[O_\lambda, u)$ with T, and $t = d(u, O_\lambda)$.

Slices. Let λ and λ' be two elements of Λ with a nonempty intersection $I \subset T$. By the hypothesis of small cancellation, I is a segment $[x_1, x_2]$ of $T \subset \widehat{T}$ of length at most $l_{\max}$. Following an idea of Gromov [25], we now identify elements $[\lambda, x, t]$ and $[\lambda', x', t']$ if $x = x' \in I$, $t = t'$ and the unoriented angles $\angle_{x_1}(O_\lambda, u)$ and $\angle_{x_2}(O_\lambda, u)$ are greater than $\theta = \frac{\pi}{2} - \pi\frac{l_{\max}}{R_{\min}}$ (θ is the angle $\angle_{x_1}(O_\lambda, x_2)$ when the segment $[x_1, x_2]$ has exactly $l_{\max}$ edges). This amounts to identifying *slices* of cones as indicated in the following picture.

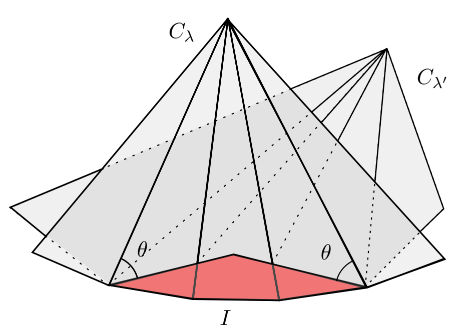

Figure III.2 - A slice identification.

Lemma III.3.1. *We have $\theta > \frac{\pi}{3}$. Furthermore, a point $[\lambda, x, t]$ of C_λ can be identified with a point in a different cone only if $t \geq r\sin(\theta) > \frac{\sqrt{3}}{2}r$.*

Proof. Since $\theta = \frac{\pi}{2} - \pi\frac{l_{\max}}{R_{\min}}$, the first part of the lemma follows from the small cancellation condition $\frac{l_{\max}}{R_{\min}} < \frac{1}{6}$. The second part is an elementary application of triangle geometry. $\square$

Let Z be the space obtained from $\widehat{T}$ by making such identifications for every $\lambda, \lambda' \in \Lambda$ with a nontrivial intersection and let Y be the quotient complex $G\backslash Z$. This space can be seen as the graph $G\backslash T$ with a collection of polygons attached and partially glued together along slices, a polygon corresponding to the image of a cone of $\widehat{T}$.

Let U'_λ (resp. U''_λ) be the polygonal neighbourhood of O_λ in $\widehat{T}$ obtained by taking the image of C_λ under the homothety of centre O_λ and ratio $\frac{1}{2}$ (resp. $\frac{1}{4}$). We define U_λ as the subcomplex obtained from U'_λ by deleting the interior of U''_λ. Up to making simplicial subdivisions, we can assume that the various U_λ, U'_λ and U''_λ are subcomplexes of $\widehat{T}$. We identify them with their images in Z.

Let g be an element of $\mathcal{R}$. Since g acts hyperbolically on T, we can write $g = h^{n_g}$ where $n_g \geq 1$ and $h \in G$ is not a proper power of an element of G. Note that h also acts hyperbolically on T, and has the same axis as g. Let $\lambda \in \Lambda$ be the element corresponding to that axis. Note that the action of the subgroup generated by h on its axis yields an action of $\mathbb{Z} = \langle h \rangle$ on U'_λ by simplicial isometries.

Let P_λ be the quotient of U'_λ under the action of $\langle g \rangle$. This is a regular polygon with $l(g) = n_g \cdot l(h)$ edges. Note that there is an action by isometries of the cyclic group $\mathbb{Z}/n_g\mathbb{Z}$ on P_λ by rotation of $l(h)$ triangles. Let $\beta_g : \mathbb{Z} \to \mathbb{Z}/n_g\mathbb{Z}$ the canonical projection. Then there is a β_g-equivariant local isometry $U_\lambda \to P_\lambda$.

We define U as the subcomplex obtained from Z by deleting the interiors of all the subcomplexes U''_λ. This subcomplex comes equipped with an action of G by simplicial isometries. Note that there is an isometric embedding $U_\lambda \hookrightarrow U$. As no non-trivial element of G fixes the axis λ, simplices in the image of the embedding $U_\lambda \hookrightarrow U$ (green region in Figure III.3) have trivial pointwise stabilisers. Let $\alpha_g : \mathbb{Z} \to G$ be the morphism sending 1 to $h \in G$. Then the isometric embedding $U_\lambda \hookrightarrow U$ is α_g-equivariant.

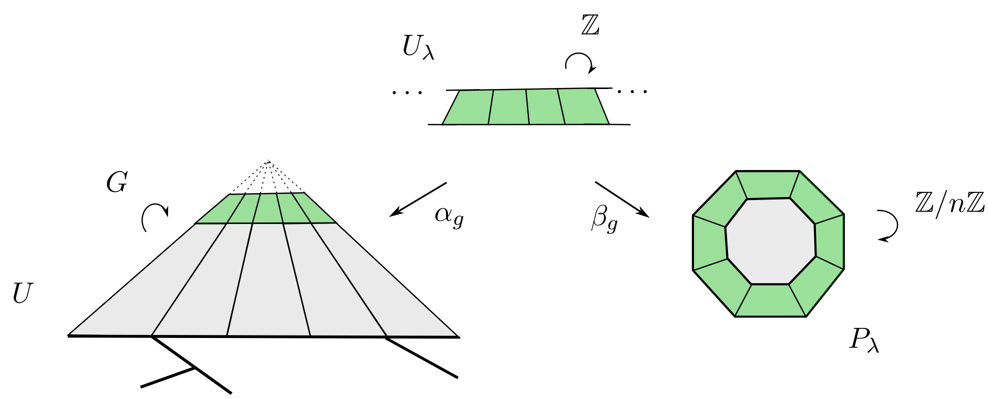

Figure III.3.

We also define the following finite complexes:

- Let V be the quotient of U by the action of G.
- Let Q_g be the quotient of P_λ by the action of $\mathbb{Z}/n_g\mathbb{Z}$.
- Let V_g be the quotient of U_λ by the action of $\mathbb{Z} = \langle h \rangle$.

Note that the α_g-equivariant embedding $U_\lambda \hookrightarrow U$ yields an isometric embedding $V_g \hookrightarrow V$. Moreover, the β_g-equivariant local isometry $U_\lambda \to P_\lambda$ yields an isometric embedding $V_g \hookrightarrow Q_g$. The finite complex obtained from the disjoint union of V and the various complexes Q_g by identifying the embedded copies $V_g \hookrightarrow V$ and $V_g \hookrightarrow Q_g$ is naturally isometric to the quotient Y; we will thus think of the complexes V, V_g, Q_g as subcomplexes of Y.

The following result will be proved in Section 3.4 by studying links of points of U.

Proposition III.3.2. *The simplicial complex U is CAT(0).*

We have the following:

Proposition III.3.3. *The simplicial complex P_λ is CAT(0).*

Proof. The link of the apex of P_λ is a loop of length $l(g)\frac{2\pi}{R_{\min}} \geq 2\pi$, so the result follows from Gromov's criterion I.2.10. □

Using the results of the previous sections, we can thus amalgamate all these actions to get a complex of groups $G(\mathcal{Y})$ over Y.

Theorem III.3.4. *The complex of groups $G(\mathcal{Y})$ is non-positively curved, hence developable, and its fundamental group is isomorphic to $G/\ll \mathcal{R} \gg$.*

Proof. The complex Y is covered by the interiors of V and the various subcomplexes Q_g. As U and P_λ are CAT(0) by III.3.2 and III.3.3, the complex of groups $G(\mathcal{Y})$ is non-positively curved, hence developable by Theorem III.1.6. To compute the fundamental group of $G(\mathcal{Y})$ we can assume that $\mathcal{R}$ is reduced to a single element $g = h^{n_g}$ (with the same notations as before), the general case following in the same way. It follows from the Van Kampen theorem III.1.4 that the fundamental group of $G(\mathcal{Y})$ is isomorphic to the amalgamated product $G *_{\mathbb{Z}} \mathbb{Z}/n_g\mathbb{Z}$, where the morphism $\alpha_g : \mathbb{Z} \to G$ sends 1 to $h \in G$, and the morphism $\beta_g : \mathbb{Z} \to \mathbb{Z}/n_g\mathbb{Z}$ is the canonical projection. Thus this group is isomorphic to $G/\ll h^{n_g} \gg$, and the result follows. □

This theorem implies the following corollary, which is well-known for ordinary small cancellation over free products with amalgamation or HNN extensions (see [34]):

Corollary III.3.5. *The quotient map $G \to {}^{G}/_{\ll \mathcal{R} \gg}$ embeds each local group of G.* □

Proof. Let $G(\mathcal{V})$ be the complex of groups over V associated to the action of G on U. By construction, this complex of groups is the restriction of $G(\mathcal{Y})$ to the subcomplex V, that is, there exists a morphism of complexes of groups $F = (F_\sigma, F(a)) : G(\mathcal{V}) \to G(\mathcal{Y})$ over the inclusion $V \hookrightarrow U$ such that each local map $F_\sigma : G_\sigma \to G_\sigma$ is the identity and all the elements $F(a)$ are trivial. The morphism F induces a map $\pi_1(G(\mathcal{V}), v_0) \to \pi_1(G(\mathcal{Y}), v_0)$ which is conjugated to $G \to {}^{G}/_{\ll \mathcal{R} \gg}$. As $G(\mathcal{Y})$ is developable, the maps $G_\sigma \to {}^{G}/_{\ll \mathcal{R} \gg}$ are injective by I.5.12, hence the result. □

Recall that a torsion element of a group acting by simplicial isometries without inversion on a CAT(0) space necessarily fixes a vertex (see [9]). Hence the CAT(0) geometry of the universal cover of $G(\mathcal{Y})$ yields a geometric proof of the following result, which is well-known for ordinary small cancellations over free products (see [34] p. 281):

Corollary III.3.6. *Let g be a torsion element in ${}^{G}/_{\ll \mathcal{R} \gg}$, then either*

(*i*) *g is the projection of a torsion element in a local factor of G, or*

(*ii*) *g is conjugate to a power of an element of $\mathcal{R}$.* □

III.3.2 A more tractable complex of groups.

Gluing slices together was used to prove that the complex of groups $G(\mathcal{Y})$ is non-positively curved. Now that we know it to be developable, we modify the construction so as to get a complex of groups that is easier to describe.

Let X be the universal covering of $G(\mathcal{Y})$ and $\widetilde{\Gamma}$ the preimage of Γ under the projection $X \to Y$. The complex X can be thought as obtained in the following way. Recall that Y is obtained from Γ by attaching to it a bunch of polygons and identifying slices of such polygons. For a polygon corresponding to the element $g \in \mathcal{R}$, any connected component of the preimage of its interior is the interior of a polygon of X. Such polygons of X are glued together according to the same slice identifications procedure.

Let P be a polygon of X and let U_P be the polygonal neighbourhood of its apex which is the image of P by the homthety of ratio $\frac{2}{3}$ centred at the apex (green region in Figure III.4). We now collapse radially the complement of U_P in P (grey region in Figure III.4), simultaneously for every polygon P of X.

Let X' be the space obtained after such collapses. This space is topologically the graph $\widetilde{\Gamma}$ with a bunch of polygons glued to it. Identifying slices in X' yields an equivariant map $X' \to X$. The action of ${}^{G}/_{\ll \mathcal{R} \gg}$ on X yields an action on X' and we denote by Y' the quotient space. Note that Y' is obtained from Y by applying the same collapsing procedure. It is the graph Γ with a collection of polygons attached to it. As this can be done without loss of generality, we will consider for the remaining of this section that this collection is reduced to a single polygon, so as to lighten notations.

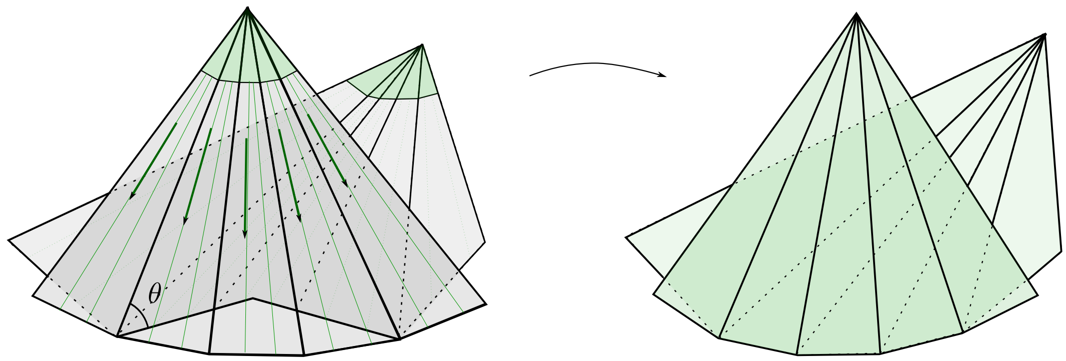

Figure III.4 - Radial collapsing.

The action of $G/\ll \mathcal{R} \gg$ on X' yields a complex of groups over Y'. In order to describe it, we first describe the complex of groups associated to the action of G on $\widehat{T}$.

Choose a basepoint v_0 of Γ which is a vertex. Let τ be a maximal tree in the first barycentric subdivision of Γ. Choose a lift $\widetilde{\tau}$ of τ to T and let T_0 be the minimal subtree of T containing $\widetilde{\tau}$. This yields a fundamental domain for the action of G on T. Let $\widetilde{v_0}$ be the unique lift of v_0 in $\widetilde{\tau}$. Let P be the polygon attached to Γ. Such a polygon corresponds to an element $g \in \mathcal{R}$. Since g acts hyperbolically on T, we can write $g = h^{n_g}$ where $n_g \geq 1$ and $h \in G$ is not a proper power of an element of G. Note that h also acts hyperbolically on T, and has the same axis as g. Up to taking a conjugate, we can assume that the axis of h meets $\widetilde{\tau}$. Let $\widetilde{\gamma}$ be the edge-path associated to the geodesic segment between $\widetilde{v_0}$ and the projection v of $\widetilde{v_0}$ on $A(g)$, let γ be its projection on Γ and γ^{-1} the reverse edge-path. We can thus write h as a $G(\Gamma)$-loop $h = \gamma g_0 e_1 g_1 \dots e_n \gamma^{-1}$ in such a way that:

- the geodesic segment $[v, hv]$ is the union $\bigcup_{1 \leq i \leq n} \gamma g_0 e_1 g_1 \dots e_i$,
- the axis $A(g)$ is the union $\bigcup_{m \in \mathbb{Z}} h^m [v, hv]$.

Note that the polygon P is attached to Γ along the edge-path $e_1 \dots e_n$, which yields a labelling on the boundary loop of P. Let $u_0, \dots, u_n$ be vertices of Γ such that $e_i = [v_{i-1}, v_i]$ for each $i = 1, \dots, n$. Let σ_i be the triangle of P whose boundary edge is labelled e_i, and a_i the edge of P between s and the boundary vertex labelled u_i.

We first describe the complex of groups associated to the action of G on $\widehat{T}$. In order to do that, we first associate to each simplex of Y a lift in $\widehat{T}$ as follows:

- We associate to the centre s of P the apex t of the cone over $A(h)$.
- We associate to a vertex of Γ its unique lift contained in $\widetilde{\tau}$.
- We associate to an edge e of Γ its unique lift $\widetilde{e}$ contained in T_0.
- We associate to the triangle σ_i its unique $\widetilde{\sigma_i}$ lift that contains an edge of $[v, h.v]$

- We associate to the edge a_i its unique lift $\widetilde{a_i}$ that is contained in $\widetilde{\sigma_{i+1}}$.

Note that the edge of $\widetilde{\sigma_i}$ contained in T is $\gamma g_0 e_1 g_1 \dots e_i =: f_i$. Let $w_0, \dots, w_n$ be vertices of T such that $f_i = [w_{i-1}, w_i]$ for each $i = 1, \dots, n$. Let $\widetilde{\gamma_i}$ be the edge-path associated to the geodesic segment of T_0 between $\widetilde{v_0}$ and the initial vertex of $\widetilde{e_i}$, let γ_i its projection to Γ and γ_i^{-1} the reverse edge-path. Then $k_i := \gamma g_0 e_1 g_1 \dots e_i \gamma_i^{-1}$ defines an element of G which sends the lift $\widetilde{e_i} = \gamma_i e_i \subset T_0$ of e_i to $\gamma g_0 e_1 g_1 \dots e_i$.

Let $K \subset T$ be the subcomplex $T_0 \cup (\bigcup_{1 \leq i \leq n} \widetilde{\sigma_i})$. For every simplex σ of K, we now define an element k_σ sending σ to the chosen lift of the image of σ in Γ:

- For $i = 1, \dots, n$, set $k_{f_i} = k_i^{-1}$.
- For $i = 0, \dots, n-1$, set $k_{w_i} = k_i^{-1}$.
- Set $k_{w_n} = h^{-1}$.
- Set $k_{h.\widetilde{a_0}} = h^{-1}$.
- For each remaining simplex σ of K, choose an arbitrary element k_σ sending σ to the lift of the image of σ in Γ.

With this set of elements, it is now possible to construct the complex of groups over Y' associated to this action. Notice already that since no non-trivial element fixes the axis $A(h)$ and h is not a proper power, the stabiliser of $A(h)$ is the subgroup generated by h, and the stabiliser of any edge or triangle of P that is not contained in Γ vanishes. We get the following list of twisting elements:

- $g_{(u_0,e_1),(e_1,\sigma_1)} = g_0^{-1}$,
- $g_{(u_{i-1},e_i),(e_i,\sigma_i)} = h_i^{-1} h_{i-1} = \gamma_i g_i^{-1} \gamma_i^{-1}$,
- $g_{(s,a_0),(a_0,\sigma_n)} = h^{-1}$,
- all the other twisting elements vanish.

This yields the following complex of groups over Y':

- The local groups and maps for vertices and edges of Γ are the same as the ones in the graph of groups over Γ.
- The local group at the centre of the polygon P is $G_s =< h >$.
- All the other local groups and maps are trivial.
- $g_{(u_0,e_1),(e_1,\sigma_1)} = g_0^{-1}$,
- $g_{(u_{i-1},e_i),(e_i,\sigma_i)} = g_i^{-1}$,

- $g_{(s,a_0),(a_0,\sigma_n)} = h^{-1}$,
- all the other twisting elements vanish.

We now turn to the complex of groups $G'(\mathcal{Y}')$ over Y' associated to the action of $G/\ll \mathcal{R} \gg$ on X. By construction, this complex of groups can be obtained as an amalgamation of the complexes of groups induced by the action of G on $\widehat{T}$ with the interior of the various U'_λ deleted, and the action of $\mathbb{Z}/n_g\mathbb{Z}$ on P_λ. This yields the following complex of groups:

- The local groups and maps for vertices and edges of Γ are the same as the ones in the graph of groups over Γ.
- The local group for the centre of the polygon is $G_s = \mathbb{Z}/n_g\mathbb{Z}$.
- All the other local groups and maps are trivial.
- $g_{(u_0,e_1),(e_1,\sigma_1)} = g_0^{-1}$,
- $g_{(u_{i-1},e_i),(e_i,\sigma_i)} = g_i^{-1}$,
- $g_{(s,a_0),(a_0,\sigma_n)}$ is the image of -1 under the canonical map $\mathbb{Z} \to \mathbb{Z}/n_g\mathbb{Z}$.
- all the other twisting elements vanish.

In order to construct a compatible complex of classifying spaces, we replace $G'(\mathcal{Y}')$ by the complex of groups $G(\mathcal{Y}')$ defined as follows:

- The local groups and maps for vertices and edges of Γ are the same as the ones in the graph of groups over Γ.
- The local group the centre of the polygon is $G_s = \mathbb{Z}/n_g\mathbb{Z}$.
- All the other local groups and maps are trivial.
- $g_{(u_0,a_0),(a_0,\sigma_1)} = g_0$,
- $g_{(u_{i-1},a_{i-1}),(a_{i-1},\sigma_i)} = g_i$,
- $g_{(s,a_1),(a_1,\sigma_1)}$ is the image of -1 under the canonical map $\mathbb{Z} \to \mathbb{Z}/n_g\mathbb{Z}$.
- all the other twisting elements vanish.

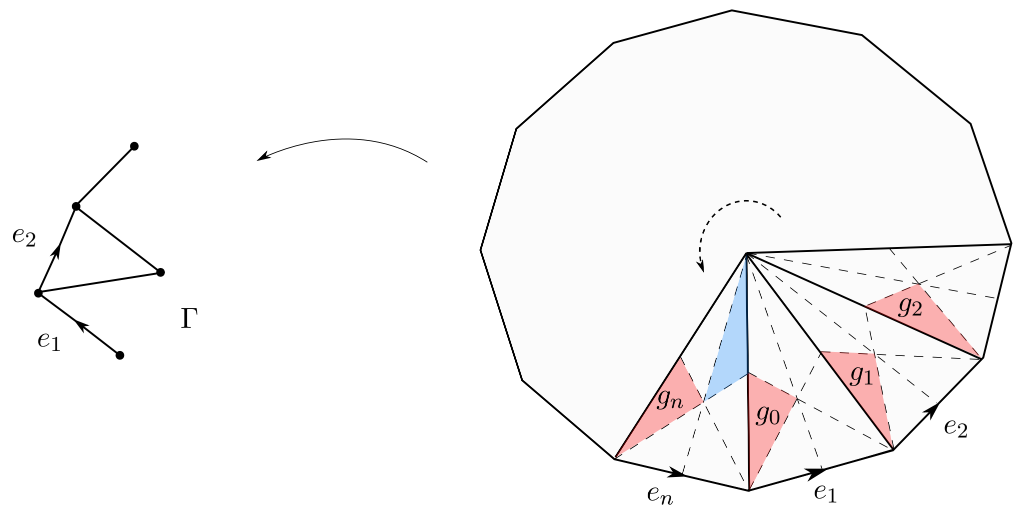

Figure III.5 - The complex of groups $G(\mathcal{Y}')$.

Note that there is an isomorphism of complexes of groups $(F_\sigma, F(a)) : G'(\mathcal{Y}') \to G(\mathcal{Y}')$ over the identity of Y':

- For each simplex σ, the local map F_σ is the identity of G_σ
- For each $i = 1, \ldots, n$, $F((u_{i-1}, \sigma_i)) = g_i$
- All the other elements $F(a)$ are trivial.

In particular, the fundamental group of $G(\mathcal{Y}')$ is ${}^{G}/_{\ll \mathcal{R} \gg}$.

III.3.3 A model of classifying space.

From now on, we assume that we are under the hypotheses of Theorem III.0.4. With the description in the previous section of a developable complex of groups $G(\mathcal{Y}')$ with fundamental group ${}^{G}/_{\ll \mathcal{R} \gg}$, it is possible to define a complex of classifying spaces compatible with $G(\mathcal{Y}')$.

Lemma III.3.7. *There exists a graph of pointed classifying spaces compatible with $G(\Gamma)$.*

Proof. For each edge e of Γ, choose an arbitrary basepoint b_e of the fibre EG_e. Let EG'_e be the CW-complex obtained by coning-off every G_e-translate of b_e (that is, EG'_e is the mapping cone of the obvious map $G_e \times \{b_e\} \to EG_e$), and b'_e be the apex corresponding to the identity element of G_e. Then the space EG'_e is a cocompact model of classifying space for G_e.

For every vertex v of Γ, consider the (finite) set of images $\phi_{v,e}(b_e) \in EG_v$, where e ranges over the set of edges containing v, and choose a compact embedded tree $K_v \subset EG_v$

containing all these images. Let EG'_v be the CW-complex obtained by coning-off every G_v-translate of K_v (that is, EG'_v is the mapping cone of the obvious map $G_v \times K_v \to EG_v$), and b'_v be the apex corresponding to the identity element of G_v. Then the space EG'_v is a cocompact model of classifying space for G_v and the $\psi_{v,e}$-equivariant map $\phi_{v,e} : EG_e \to EG_v$ extends to a $\psi_{v,e}$-equivariant map $\phi'_{v,e} : EG'_e \to EG'_v$ sending b'_e to b'_v. Thus, the collection $\big((EG'_v, b'_v), (EG'_e, b'_e), \phi'_{v,e}\big)$ defines a graph of pointed classifying spaces compatible with $G(\Gamma)$. □

So as to lighten notations, we consider from now on a graph of pointed classifying space $EG(\Gamma) = \big((EG_v, b_v), (EG_e, b_e), \phi_{v,e}\big)$ compatible with $G(\Gamma)$. We define a complex of classifying spaces $EG(\mathcal{Y}')$ compatible with $G(\mathcal{Y}')$ as follows:

- The restriction of $EG(\mathcal{Y}')$ to Γ is just $EG(\Gamma)$.
- We associate to each triangle σ_i of P and to the apex s of P a classifying space reduced to a point. We associate to each edge a_i of P a copy EG_{a_i} of the unit interval $[0, 1]$.
- We define the map $\phi_{v_i,a_i} : EG_{a_i} \to EG_{v_i}$ as follows: ϕ_{v_i,a_i} sends 0 to the basepoint b_{v_i} of EG_{v_i} and sends 1 to $g_i.b_{v_i}$. Moreover, ϕ_{v_i,a_i} sends the unit interval to a path from b_{v_i} to $g_i.b_{v_i}$.
- The map $\phi_{a_{i-1},\sigma_i} : EG_{\sigma_i} \to EG_{a_{i-1}}$ sends the basepoint of EG_{σ_i} to $1 \in [0, 1]$.
- The map $\phi_{a_i,\sigma_i} : EG_{\sigma_i} \to EG_{a_i}$ sends the basepoint of EG_{σ_i} to $0 \in [0, 1]$.
- The map $\phi_{e_i,\sigma_i} : EG_{\sigma_i} \to EG_{e_i}$ sends the basepoint of EG_{σ_i} to the basepoint of EG_{e_i}.
- All the remaining local maps are trivial.

Theorem III.3.8. *If all the local groups of the graph of groups $G(\Gamma)$ admit cocompact models of classifying spaces then so does $G/\ll \mathcal{R} \gg$.*

Proof. It is straightforward to check that the previous complex of classifying spaces is compatible with the complex of group $G(\mathcal{Y}')$. If $G/\ll \mathcal{R} \gg$ contains torsion, let H be a non-trivial finite subgroup of $G/\ll \mathcal{R} \gg$. First notice that the fixed point set $(X')^H$ is contained in the graph $\widetilde{\Gamma}$. We claim that it does not contain any non-trivial loop. If this was not the case, by considering the image of a non-trivial loop of $(X')^H$ in the CAT(0) space X under the equivariant projection $X' \to X$, Proposition I.1.6 would imply that there exists a polygon of X with a non-trivial pointwise stabiliser, which is absurd. Thus $(X')^H$ is contractible and one can use the remark following Theorem II.2.3 to conclude. □

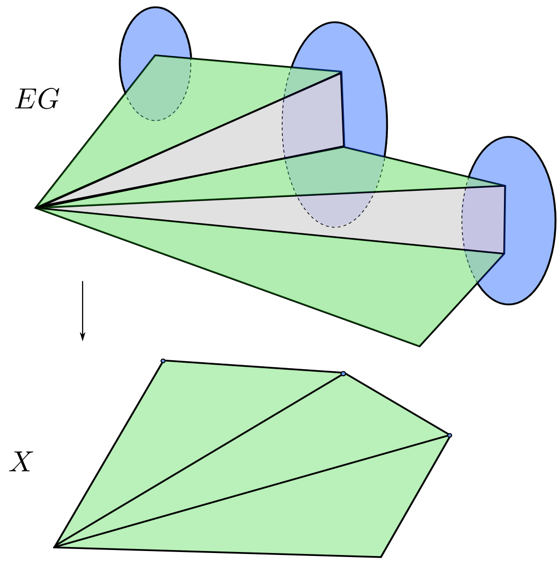

Figure III.6 A portion of the realisation of the associated complex of spaces.

III.4 The geometry of Z.

We now prove that U is CAT(0). Note that U being homotopy equivalent to the Bass-Serre tree U, it is contractible, hence we only have to prove that it is locally CAT(0). Since U is a 2-dimensional complex, it is enough to prove, by the Gromov criterion I.2.10, that injective loops in links of points of U have length at least 2π. As this condition is preserved by taking subcomplexes, it is thus enough to prove that the space Z itself is CAT(0).

There are three types of points in Z: apeces of cones of Z, points in the Basse-Serre tree T and points in the interior of a cone.

Apex of a cone. Each apex of a cone of Z has a link simplicially isomorphic to a bi-infinite line, hence the Gromov criterion I.2.10 is satisfied.

Point in the interior of a cone. Let u be a point that is in the interior of a cone but is not an apex. A neighbourhood of u in Z is obtained from neighbourhoods of u in the various cones containing it by gluing them together in an appropriate way. Let Λ_u be the set of λ such that C_λ contains u and let $\lambda \in \Lambda_u$. A polygonal neighbourhood of $u = [\lambda, x, t]$ in $C_\lambda \in \Lambda$ is obtained as follows. Consider four small segments $a_\lambda, b_\lambda, c_\lambda, d_\lambda$ issuing from u with an unoriented angle $\theta(t)$ and $\pi - \theta(t)$ with the ray $[0_\lambda, u)$, where $\theta(t) = \arcsin(\frac{r}{t}\sin\theta)$ is the angle indicated in Figure III.7 (note that we have $\theta(t) \geq \theta \geq \frac{\pi}{3}$). We use these segments to define an arbitrarily small polygonal neighbourhood of u as indicated in the following picture, along with the link of u with respect to that polygonal neighbourhood:

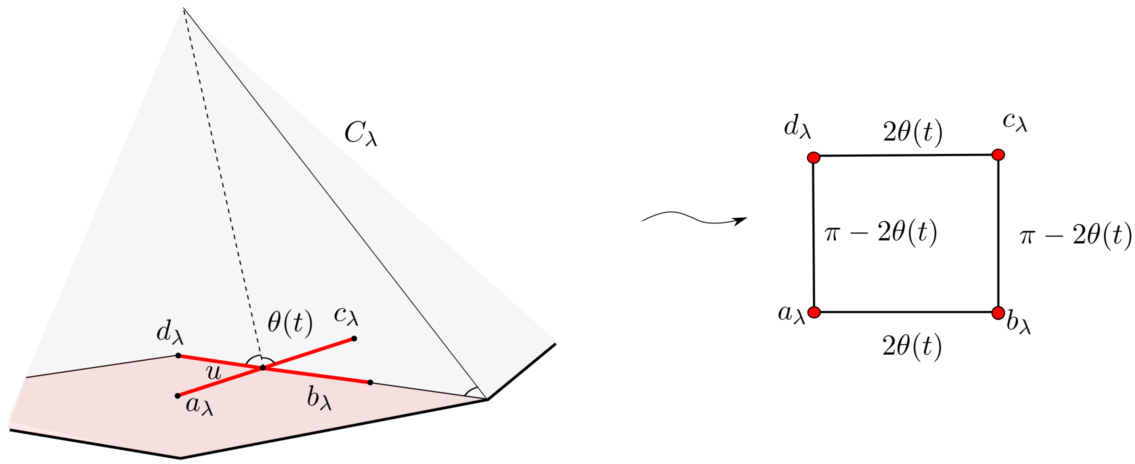

Figure III.7 - The link $\mathrm{lk}(u, C_\lambda)$.

We now explain how these graphs are glued together under the identifications defining Z. Let $\lambda, \mu \in \Lambda_u$ and let us look at u inside C_λ. Let us call $C_{\lambda,\mu} \subset C_\lambda$ the slice along which C_λ and C_μ were glued. If u belongs to the interior of $C_{\lambda,\mu}$, then the two graphs are identified in the obvious way. If u belongs to the boundary of $C_{\lambda,\mu}$, then the two graphs are glued as follows:

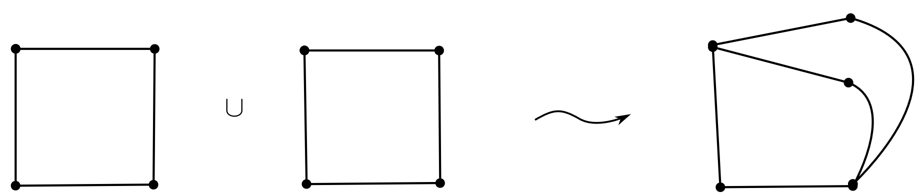

Figure III.8- Some link identifications.

Thus, the link of u in Z is a graph without loop or double edge, and which has four types of vertices: a vertex a corresponding to edges a_λ after identification, a vertex b corresponding to edges b_λ after identification, vertices $c_1, c_2, \dots$ corresponding to edges c_λ, which are of valence at least 2, and vertices $d_1, d_2, \dots$ corresponding to edges d_λ., which are of valence at least 2. Moreover, the following holds:

- There is exactly one edge between a and b (of length $2\theta(t)$).
- There is exactly one edge between a and each d_i (of length $\pi - 2\theta(t)$) and exactly one edge between b and each c_i (of length $\pi - 2\theta(t)$).
- The graph is bipartite with respect to the decomposition of the set of vertices into the sets $\{a\} \cup \{c_1, c_2, \dots\}$ and $\{b\} \cup \{d_1, d_2 \dots\}$.

- Edges of the form $[c_i, d_j]$ are of length $2\theta(t)$.

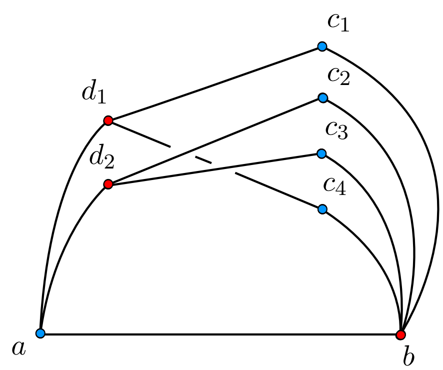

Figure III.9 - The link $\mathrm{lk}(u, Z)$.

Lemma III.4.1. *An injective loop in the link lk(u, Z) has length at least* 2π*.*

Proof. As the link is a bipartite graph, an injective loop contains an even number of edges. Since the graph has no double edge, the loop is made of at least four edges. Now since the subgraph with all but one of the edges of the form $[a, b]$ or $[c_i, d_j]$ removed is a tree, the loop must contains two edges of the form $[a, b]$ or $[c_i, d_j]$. As these edges have length $2\theta(t) \geq \frac{2\pi}{3}$ and the remaining ones have length $\pi - 2\theta(t) \leq \frac{\pi}{3}$, such a loop has length at least $2.2\theta(t) + 2(\pi - 2\theta(t)) = 2\pi$. $\square$

Points in the Bass-Serre tree. Let v be a point in the Bass-Serre tree T. If v is not a vertex of T, then a neighbourhood of v in Z is given by choosing a small neighbourhood of v in any cone containing it, hence such points have CAT(0) neighbourhoods. Now let v be a vertex of T. Let Λ_v be the set of λ such that C_λ contains v and let $\lambda \in \Lambda_v$. A polygonal neighbourhood of v in C_λ is obtained as follows. Let a_λ, a'_λ be the two edges of T issuing from v that are contained in C_λ. Let c_λ be the radius $[O_\lambda, v]$. Let b_λ (resp. b'_λ) be the segment of C_λ issuing from v that makes an unoriented angle $\pi - \theta(t)$ with the ray a_λ (resp. a'_λ). We use these segments to define an arbitrarily small polygonal neighbourhood of u as indicated in the following pictrue:

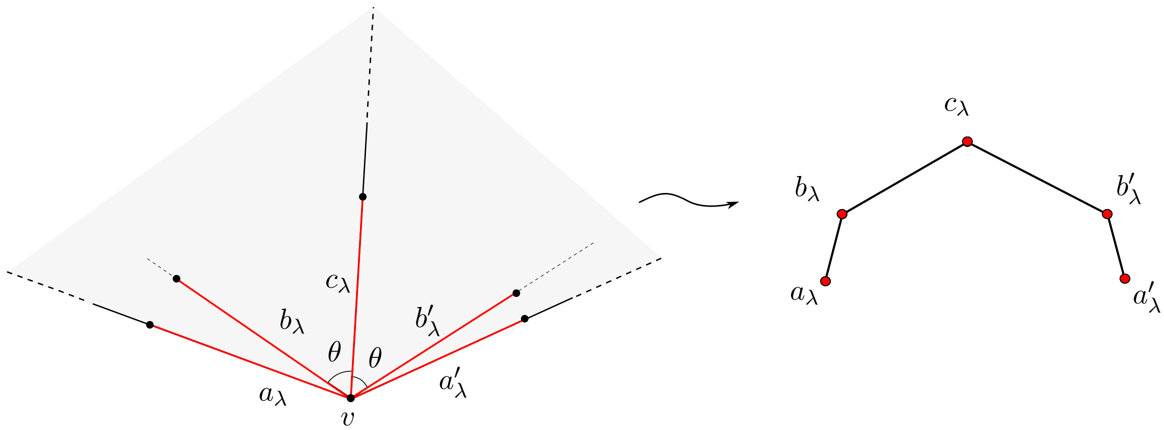

Figure III.10 - The link $\mathrm{lk}(v, C_\lambda)$.

We now look at how the links $\mathrm{lk}(v, C_\lambda)$ and $\mathrm{lk}(v, C_{\lambda'})$ are glued together. Let $\lambda, \mu \in \Lambda_v$ and let us look at v inside C_λ. If v belongs to the interior of $C_{\lambda,\mu}$, then the two links are identified in the obvious way. If v belongs to the boundary of $C_{\lambda,\mu}$, then the two links are glued along a common edge as follows:

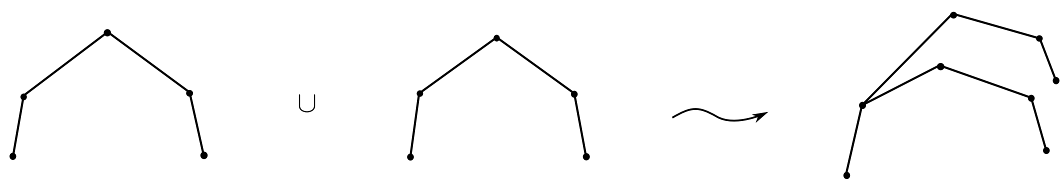

Figure III.11 - Some link identifications.

Thus, the link $\mathrm{lk}(v, Z)$ is a graph with no double edge or loop and which has three types of vertices:

- Vertices $a_1, a_2, \ldots$ (type A) corresponding to edges of T. These vertices are of valence 1.

- Vertices $b_1, b_2, \ldots$ (type B) corresponding to segments b_λ, b'_λ, $\lambda \in \Lambda_v$. These vertices are of valence at least 2.

- Vertices $c_1, c_2, \ldots$ (type C) corresponding to edges c_λ, $\lambda \in \Lambda_v$. These vertices are of valence 2.

Furthermore, $\mathrm{lk}(v, Z)$ is a tripartite graph with respect to the partition of the set of its vertices into the aforementioned three types A, B and C.

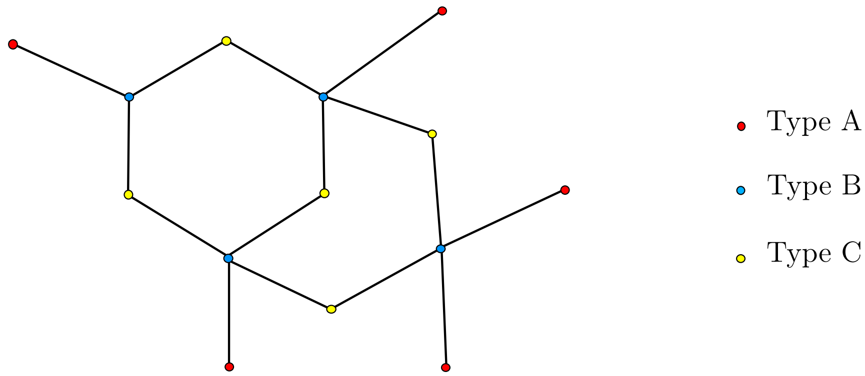

Figure III.12 - The link $\mathrm{lk}(v, Z)$.

Lemma III.4.2. *An injective loop in the link lk(v, Z) has length at least 2π.*

Proof. Let γ be an injective loop in $\mathrm{lk}(v, Z)$. Since type A vertices have valence 1, γ only meets type B and type C vertices. Moreover, γ is a bipartite graph for the induced colouring, hence it has an even number of edges. As there is no double edge, γ has at least four edges.

We prove by contradiction that it cannot contain exactly four edges. Indeed, γ would then contain two type C vertices corresponding to edges $c_\lambda, c_{\lambda'}$ $(\lambda, \lambda' \in \Lambda_v)$ and the remaining two vertices would thus correspond to the associated edges $b_\lambda, b'_\lambda, b_{\lambda'}, b'_{\lambda'}$ after identification. Consequently, γ would be contained in the image of $\mathrm{lk}(v, C_\lambda) \cup \mathrm{lk}(v, C_{\lambda'})$ after identification, but the above discussion shows that this image does not contain an injective cycle (see Figure III.11).

Thus, γ contains at least six edges, all of whose being between a type B vertex and a type C vertex. As the length of such an edge is $\theta > \frac{\pi}{3}$, the length of γ is at least $6\theta > 2\pi$. □

Corollary III.4.3. *The complex Z is CAT(0).* □

Chapter IV

A combination theorem for boundaries of groups.

In this chapter, we give conditions under which it is possible to construct a Bestvina boundary for the fundamental group of a non-positively curved complex of groups out of such structures for its local groups.

Theorem IV.0.4 (Combination Theorem for Boundaries of Groups)**.** *Let $G(\mathcal{Y})$ be a non-positively curved complex of groups over a finite simplicial complex Y endowed with a M_κ-structure, $\kappa \leq 0$. Let G be the fundamental group of $G(\mathcal{Y})$ and X be a universal covering of $G(\mathcal{Y})$. Suppose that the following global condition holds:*

(i) The action of G on X is acylindrical, that is, there exists a uniform bound on the diameter of a subset of X with infinite pointwise stabiliser.

Further assume that there is an $E\mathcal{Z}$-complex of classifying spaces compatible with $G(\mathcal{Y})$ that satisfies each of the following local conditions:

(ii) the limit set property: for every simplex σ of Y, and every pair of subgroups H_1, H_2 in the family $\mathcal{F}_\sigma = \left\{\bigcap_{i=1}^n g_i G_{\sigma_i} g_i^{-1} |\ g_1, \ldots, g_n \in G_\sigma, \sigma_1 \supset \sigma, \ldots, \sigma_n \supset \sigma, n \in \mathbb{N}\right\}$, we have $\Lambda H_1 \cap \Lambda H_2 = \Lambda(H_1 \cap H_2) \subset \partial G_\sigma$.

(iii) the convergence property: for every pair of simplices $\sigma \subset \sigma'$ in Y and every sequence (g_n) of G_σ whose projection is injective in $G_\sigma/G_{\sigma'}$, there exists a subsequence such that $(g_{\varphi(n)}\overline{EG_{\sigma'}})$ uniformly converges to a point in $\overline{EG_\sigma}$.

(iv) the finite height property: for every pair of simplices $\sigma \subset \sigma'$ of Y, $G_{\sigma'}$ has finite height in G_σ (see [22]), that is, there exist an upper bound on the number of distinct cosets $\gamma_1 G_{\sigma'}, \ldots, \gamma_n G_{\sigma'} \in G_\sigma/G_{\sigma'}$ such that the intersection $\gamma_1 G_{\sigma'} \gamma_1^{-1} \cap \ldots \cap \gamma_n G_{\sigma'} \gamma_n^{-1}$ is infinite.

Then G admits an $E\mathcal{Z}$-structure $(\overline{EG}, \partial G)$. Furthermore, the following properties hold:

(ii') *For every simplex σ of Y, the map $\overline{EG_\sigma} \to \overline{EG}$ realises an equivariant embedding from ∂G_σ to $\Lambda G_\sigma \subset \partial G$. Moreover, for every pair H_1, H_2 of subgroups in the family $\mathcal{F} = \left\{\bigcap_{i=1}^n g_i G_{\sigma_i} g_i^{-1} |\ g_1, \ldots, g_n \in G,\ \sigma_1, \ldots, \sigma_n \in S(Y),\ n \in \mathbb{N}\right\}$, we have $\Lambda H_1 \cap \Lambda H_2 = \Lambda(H_1 \cap H_2) \subset \partial G$.*

(iii') *For every simplex σ of Y, the embedding $\overline{EG_\sigma} \hookrightarrow \overline{EG}$ satisfies the convergence property.*

(iv') *For every simplex σ of Y, the local group G_σ has finite height in G.*

The chapter is organised as follows. In Section 1, we continue our study of geodesics in M_κ-complexes. We define the boundary ∂G of G and the compactification $\overline{EG}$ of EG as sets in Section 2. In Section 3, we introduce further conditions on a complex of groups or spaces. In Section 4 we study the geometry of important subcomplexes of X, called domains, which were implicitly used to define ∂G. Section 5 is devoted to the proof of some geometric results that are used throughout the chapter. We define a topology on $\overline{EG}$ in Section 6 and we prove that it makes $\overline{EG}$ a compact metrisable space in Section 7. The proof of Theorem IV.0.4 is completed in Section 8, where the properties of ∂G are investigated.

We choose once and for all a non-positively curved complex of groups $G(\mathcal{Y})$ over a finite simplicial complex endowed with a M_κ-structure, $\kappa \leq 0$. Recall that a complex of groups consists of the data $(G_\sigma, \psi_a, g_{a,b})$ of local groups (G_σ), local maps (ψ_a) and twisting elements $(g_{a,b})$. We fix a maximal tree T in the 1-skeleton of the first barycentric subdivision of Y, which allows us to define the fundamental group $G = \pi_1(G(\mathcal{Y}), T)$ and the canonical morphism $\iota_T : G(\mathcal{Y}) \to G$ given by the collection of injections $G_\sigma \to G$. Finally, we define X as the universal covering of $G(\mathcal{Y})$ associated to ι_T. The simplicial complex X naturally inherits a M_κ-structure from that of Y and the simplicial metric d on X makes it a complete geodesic metric space, which is CAT(0) by the curvature assumption on $G(\mathcal{Y})$.

IV.1 Geodesics in M_κ-complexes.

In this section, we study the geometry of the set of geodesics of an M_κ-complex. Recall that X is assumed to be a M_κ-complex, $\kappa \leq 0$, with finitely many isometry types of simplices.

IV.1.1 The finiteness lemma.

Definition IV.1.1. For subsets K, L of X, we define $\mathrm{Geod}(K, L)$ as the set of points lying on a geodesic segment from a point of K to a point of L.

Definition IV.1.2 (Simplicial neighbourhood)**.** Let K be a subcomplex of X. The union of the closed simplices that meet K is called the *closed simplicial neighbourhood* of K, and denoted $\overline{\mathrm{N}}(K)$. The union of the open simplices whose closure meets K is called the *open simplicial neighbourhood* of K, and denoted $\mathrm{N}(K)$.

We recall the following proposition of Bridson, which follows from the Claim contained in the proof of Theorem 1.11 of [8].

Proposition IV.1.3 (containment lemma, Bridson [8])**.** *For every n there exists a constant k such that for every finite subcomplex $K \subset X$ containing at most n simplices, any geodesic path contained in the open simplicial neighbourhood of K meets at most k simplices.* □

We also recall this useful related result, which follows from Theorem 1.11 of [8].

Corollary IV.1.4 (Bridson [8])**.** *For every n there exists a constant k such that every geodesic segment of length at most n meets at most k simplices.* □

Lemma IV.1.5 (Finiteness lemma)**.** *Let X be as before. For subcomplexes $K, K' \subset X$, Geod(K, K') meets only finitely many open simplices.*

Proof. It is enough to prove the result when K and K' consist of two closed simplices σ and σ'. For every $x \in \sigma$ and every $x' \in \sigma'$, we consider the sequence of open simplices $\sigma_1, \ldots, \sigma_n$ met by the geodesic segment $[x, x']$ and set $C_{x,x'} = \sigma \cup \sigma_1 \cup \ldots \cup \sigma_n \cup \sigma'$. Note that by Corollary IV.1.4 there is a uniform k bound on the number of simplices contained in $C_{x,x'}$. Since there is only finitely many isometry types of simplices in X, there is, up to simplicial isometry fixing pointwise σ and σ', finitely many subcomplexes of the form $C_{x,x'}$. Following Bridson, we call such an equivalence class of subcomplexes a *model* (see the proof of I.7.57 in [9]).

We now claim that for every $x, y \in \sigma$ and every $x', y' \in \sigma'$ such that $C_{x,x'}$ and $C_{y,y'}$ are in the same model, we have $C_{x,x'} = C_{y,y'}$. Indeed, choose a simplicial isometry $\phi : C_{x,x'} \to C_{y,y'}$ that fixes pointwise σ and σ'. Then ϕ sends the geodesic segment $[x, x'] \subset C_{x,x'}$ to a simplicial path of the same length between $\phi(x) = x$ and $\phi(x') = x'$. As X is CAT(0), geodesic segments are unique, hence ϕ pointwise fixes $[x, x']$. We thus have $[x, x'] = \phi\big([x, x']\big) \subset C_{y,y'}$, hence $C_{x,x'} \subset C_{y,y'}$. The same reasoning applied to the geodesic segment $[y, y']$ yields $C_{y,y'} \subset C_{x,x'}$, hence $C_{x,x'} = C_{y,y'}$.

We have

$$\mathrm{Geod}(\sigma, \sigma') \subset \bigcup_{x \in \sigma, x' \in \sigma'} C_{x,x'}$$

and the previous discussion shows that this is a finite union, which concludes the proof. □

IV.1.2 Paths of simplices of uniformly bounded length.

Definition IV.1.6. A *path of simplices* is a sequence of open simplices $\sigma_1, \ldots, \sigma_n$ such that $\overline{\sigma_i} \subset \overline{\sigma_{i+1}}$ or $\overline{\sigma_{i+1}} \subset \overline{\sigma_i}$ for every $i = 1, \ldots, n-1$. Equivalently, it is a finite path in the first barycentric subdivion of X. The integer n is called the *length* of the path of simplices.

Up to rescaling the metric, we also make the following assumption:

From now on, we will assume that the distance from any simplex to the boundary of its (closed) simplicial neighbourhood is at least 1.

Here we prove the following lemma:

Lemma IV.1.7 (Short paths of simplices)**.** *For every $n \geq 1$, there exists $m \geq 1$ such that the following holds: Let K be a convex subcomplex of X and K' a connected subcomplex of X, both containing at most n simplices. Let $x, y \in K$ and $x', y' \in K'$ and assume that there exists a path in K' between x' and y' that does not meet K. Let τ, τ' be two simplices of $N(K) \setminus K$ such that the geodesic segment $[x, x']$ (resp. $[y, y']$) meets the interior of τ (resp. τ'). Then there exists a path of simplices in $N(K) \setminus K$ of length at most m between τ and τ'.*

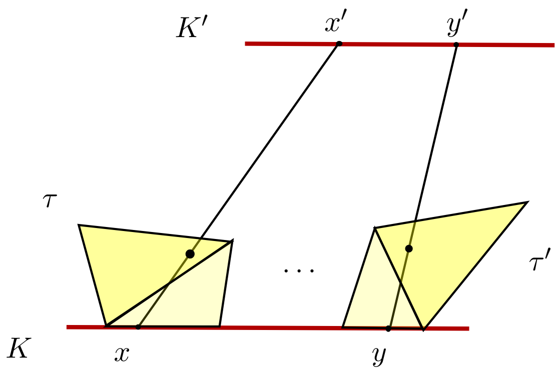

Figure IV.1

Definition IV.1.8 (I.7.8 of [9])**.** For $x \in X$, let

$$\eta(x) = \inf\{d(x, \sigma) |\ \sigma \subset \overline{\mathrm{st}}(\sigma_x),\ x \notin \sigma\}.$$

The constant is such that for every $y \in B(x, \eta(x))$, we have $\sigma_x \subset \sigma_y$.

The following lemma is a controlled version of Lemma I.7.54 in [9].

Lemma IV.1.9. *There exist constants $\eta_0 > \varepsilon_0 > 0$ such that:*

- *for every simplex σ of X, the $2\eta_0$-neighbourhood of σ is contained in the open simplicial neighbourhood of σ;*
- *for every point $x \in X$, there exists $y \in B(x, \eta_0)$ such that $B(x, \varepsilon_0) \subset B(y, \eta(y))$.*

Proof. For a simplex σ of X, let

$$\eta(\sigma) = \inf\{d(\sigma,\tau) |\ \tau \subset \overline{\mathrm{N}}(\sigma),\ \sigma \cap \tau = \varnothing\}.$$

The above set of distances is finite since X has only finitely many isometry types of simplices, thus $\eta(\sigma) > 0$. For the same reason, we can define $\eta_0 = 1/2 \cdot \min\ \eta(\sigma) > 0$, where the minimum is taken over all the simplices of X.

Now that η_0 is defined, we construct constants $\eta_1, \ldots, \eta_D$ by induction, where D is the maximal dimension of a simplex of X, as well as subets $T_0, \ldots, T_D$ of X, such that each T_k is an open neighbourhood of the k-skeleton $X^{(k)}$ of X.

Let

$$T_0 = \bigcup_{v \in V(X)} B(v, \eta_0),$$

where η_0 is as above. Suppose that $\eta_0, \ldots, \eta_k$ and $T_0, \ldots, T_k$ are defined. For each simplex $\sigma \subset X$ of dimension $k+1$, the function η (as defined in IV.1.8) is continuous on the compact set $\sigma \setminus T_k$ and does not vanish, hence is bounded below by a constant $\eta_{k+1}(\sigma) > 0$. As X has finitely many isometry types of simplices, we define $\eta_{k+1} = 1/2 \cdot \min\ \eta_{k+1}(\sigma) > 0$, where the minimum is taken over all simplices of dimension $k+1$. We can further assume that $\eta_{k+1} \leq \eta_k$. Let

$$T_{k+1} = T_k \cup \Bigg(\bigcup_{\substack{\sigma \subset X, \\ \dim\ \sigma = k+1}} \ \bigcup_{x \in \sigma \setminus T_k} B(x, \eta_{k+1}) \Bigg).$$

Finally, let $\varepsilon_0 = \eta_D$. We have $T_0 \subset \ldots \subset T_D = X$. Let $x \in X$. There exists a unique k such that $x \in T_k \setminus T_{k-1}$. For such a k, there exists $y \in X^{(k)} \setminus T_{k-1}$ with $d(x,y) \leq \eta_k$ (in particular $d(x,y) \leq \eta_0$). As $\varepsilon_0 \leq \eta_k$, we get

$$B(x, \varepsilon_0) \subset B(x, \eta_k) \subset B(y, 2\eta_k) \subset B(y, \eta_k(\sigma_y)) \subset B(y, \eta(y)),$$

which concludes the proof. □

Proof of Lemma IV.1.7. First notice that since X has only finitely many isometry types of simplices, there exists a constant l, which depends only on n and X, points $x = x_0, \ldots, x_l = y$ in K and $x' = x'_0, \ldots, x'_l = y'$ in K' such that for every k, $d(x_k, x_{k+1}) < \varepsilon_0$, $d(x'_k, x'_{k+1}) < \varepsilon_0$, x_k, x_{k+1} belong to the same simplex of K, and x'_k, x'_{k+1} belong to the same simplex of K'. For every $k = 1 \ldots, l-1$, let τ_k be a simplex of $\mathrm{N}(K) \setminus K$ whose interior meets $[x_k, x'_k]$. In order to prove Lemma IV.1.7, it is thus enough to consider the case where $d(x,y) < \varepsilon_0$, $d(x', y') < \varepsilon_0$, x, y belong to the same simplex σ of K, and x', y' belong to the same simplex σ' of K'. We treat separately two cases.

Case 1: Suppose that the geodesic segments $[x, x']$ and $[y, y']$ are both contained in the open η_0-neighbourhood of K. Recall that by definition of η_0, this implies that they are contained in the open simplicial neighbourhood of K. The geodesic segment $[x, x']$ yields a geodesic segment, contained in $\mathrm{N}(K) \setminus K$ by convexity of K, between a point in the interior of τ and x'. By Proposition IV.1.3, there exists a constant m_1 (which depends only on X and n) such that there exists a path of simplices in $\mathrm{N}(K) \setminus K$ of length at most m_1 between τ and σ'. Reasoning similarly for $[y, y']$, we get a path of simplices in $\mathrm{N}(K) \setminus K$ of length at most m_1 between τ' and σ'. We thus get a path of simplices in $\mathrm{N}(K) \setminus K$ of length at most $2m_1$ between τ and τ'.

Case 2: Suppose that the geodesic segment $[x, x']$ is not contained in the η_0-neighbourhood of K. We then choose a point u on that geodesic segment which belongs to $B(K, 2\eta_0) \setminus B(K, \eta_0)$ (such a subset is contained in $\mathrm{N}(K)$ by definition of η_0). By Lemma IV.1.9, we can choose $z \in X \setminus K$ such that $B(u, \varepsilon_0) \subset B(z, \eta(z))$. Since $d(x, y) < \varepsilon_0$ and $d(x', y') < \varepsilon_0$, the CAT(0) geometry of X implies that $[y, y']$ meets the ball $B(u, \varepsilon_0) \subset B(z, \eta(z))$ at a point v. By definition of $\eta(z)$, we thus have $\sigma_z \subset \sigma_u$ and $\sigma_z \subset \sigma_v$, which yields the path of simplices $\sigma_u, \sigma_z, \sigma_v$ in $\mathrm{N}(K) \setminus K$ between σ_u and σ_v. Now the geodesic segment $[x, x']$ (resp. $[y, y']$) yields a path of simplices in $\mathrm{N}(K) \setminus K$ (by convexity of K) of length at most m_1 between τ and σ_u (resp. between τ' and σ_v). We thus get a path of simplices in $\mathrm{N}(K) \setminus K$ of length at most $2m_1 + 1$ between τ and τ'. □

IV.2 Construction of the boundary

We now turn to the construction of a boundary of G.

Definition IV.2.1. We say that a complex of classifying spaces $EG(\mathcal{Y})$ compatible with a complex of groups $G(\mathcal{Y})$ extends to an $E\mathcal{Z}$*-complex of classifying spaces* if it satisfies the following extra conditions:

- Each fibre EG_σ is endowed with an $E\mathcal{Z}$-structure $(\overline{EG_\sigma}, \partial G_\sigma)$.
- Each local map $\phi_a : EG_{i(a)} \to EG_{t(a)}$ is an equivariant embedding and extends to an equivariant embedding $\phi_a : \overline{EG_{i(a)}} \to \overline{EG_{t(a)}}$, such that for every pair (a, b) of composable edges of $\mathcal{Y}$, we have:

$$g_{a,b} \circ \phi_{ab} = \phi_a \phi_b.$$

Definition IV.2.2. We define the space

$$\Omega(\mathcal{Y}) = \Big(G \times \coprod_{\sigma \in V(\mathcal{Y})} (\{\sigma\} \times \partial G_\sigma)\Big) / \simeq$$

where

$$\big(gg', (\{\sigma\}, \xi)\big) \simeq \big(g, (\{\sigma\}, g'\xi)\big) \text{ if } \xi \in \partial G_\sigma, g' \in G_\sigma, g \in G.$$

It should be noted here that $\{\sigma\}$ denotes a point labeled by σ and not the simplex itself. The set $\Omega(\mathcal{Y})$ comes with a natural projection to the set of simplices of X. If σ is a simplex of X, we denote by ∂G_σ the preimage of $\{\sigma\}$ under that projection. We now define

$$\partial_{Stab} G = \Omega(\mathcal{Y})/\sim$$

where $\sim$ is the equivalence relation generated by the following identifications:

$$\Big[g, \{\sigma'\}, \xi\Big] \sim \Big[gF((\sigma,\sigma'))^{-1}, \{\sigma\}, \phi_{(\sigma,\sigma')}(\xi)\Big] \text{ if } g \in G, (\sigma,\sigma') \in E(\mathcal{Y}) \text{ and } \xi \in \partial G_{\sigma'}.$$

The action of G on $G \times \coprod_{\sigma\in V(\mathcal{Y})}(\{\sigma\} \times \partial G_\sigma)$ by left multiplication on the first factor yields an action of G on $\Omega(\mathcal{Y})$ and on $\partial_{Stab}G$.

Definition IV.2.3. We define the spaces $\partial G = \partial_{Stab}G \cup \partial X$ and $\overline{EG} = EG \cup \partial G$.

Our aim is to endow $\overline{EG}$ with a topology that makes $(\overline{EG}, \partial G)$ an $E\mathcal{Z}$-structure for G.

Notation: Since the $\phi_{\sigma,\sigma'}$ are embeddings, we identify $\phi_{\sigma,\sigma'}(\overline{EG_{\sigma'}}) \subset \overline{EG_\sigma}$ with $\overline{EG_{\sigma'}}$. For instance, if U is an open subset of $\overline{EG_\sigma}$, we will simply write " we have $\overline{EG_{\sigma'}} \subset U$ in $\overline{EG_\sigma}$" instead of "we have $\phi_{\sigma,\sigma'}(\overline{EG_{\sigma'}}) \subset U$ in $\overline{EG_\sigma}$".

From now on, we assume that there is a complex of classifying spaces $EG(\mathcal{Y})$ that extends to an $E\mathcal{Z}$-complex of classifying spaces compatible with the complex of groups $G(\mathcal{Y})$.

IV.2.1 Further properties of $E\mathcal{Z}$-complexes of spaces.

In this section, we define additional properties of $E\mathcal{Z}$-complexes of spaces, which will enable us to study the properties of the equivalence relation $\sim$ previously defined.

The limit set property.

Recall that for a discrete group Γ together with an $E\mathcal{Z}$-structure $(\overline{E\Gamma}, \partial\Gamma)$ and a subgroup H, the *limit set* ΛH of H in $\partial\Gamma$ is the set $\overline{Hx} \cap \partial\Gamma$, where x is an arbitrary point of $E\Gamma$.

Definition IV.2.4 (Limit set property for an $E\mathcal{Z}$-complex of classifying spaces)**.** Let $EG(\mathcal{Y})$ be an $E\mathcal{Z}$-complex of classifying spaces compatible with the complex of groups $G(\mathcal{Y})$. By definition, we already have that for every pair of simplices $\sigma \subset \sigma'$ of Y, the map $\phi_{\sigma,\sigma'}$ is an equivariant homeomorphism from $\partial G_{\sigma'}$ to the limit set $\Lambda G_{\sigma'} \subset \partial G_\sigma$. We further say that $EG(\mathcal{Y})$ satifies the *limit set property* if in addition, we have:

- For every simplex σ of Y, and every pair of subgroups H_1, H_2 in the family $\mathcal{F}_\sigma = \left\{\bigcap_{i=1}^n g_i G_{\sigma_i} g_i^{-1} \mid g_1,\dots,g_n \in G_\sigma, \sigma_1 \supset \sigma, \dots, \sigma_n \supset \sigma, n \in \mathbb{N}\right\}$, we have $\Lambda H_1 \cap \Lambda H_2 = \Lambda(H_1 \cap H_2)$.

Remarks. (*i*) Let Γ be a hyperbolic group, and H a subgroup. Then H is quasiconvex if and only if its limit set in $\partial\Gamma$ is equivariantly homeomorphic to ∂H, by a result of Bowditch [6].
(*ii*) Let Γ be a hyperbolic group and $\partial\Gamma$ its Gromov boundary. Let H_1 and H_2 be two quasiconvex subgroups of Γ. Then $\Lambda H_1 \cap \Lambda H_2 = \Lambda(H_1 \cap H_2)$ by a result of [24].

The finite height property.

Recall that, for Γ a discrete group and H a subgroup, the *height* of H is the supremum of the set of integers $n \in \mathbb{N}$ such that there exist distinct cosets $\gamma_1 H, \ldots, \gamma_n H \in G/H$ such that the intersection $\gamma_1 H \gamma_1^{-1} \cap \ldots \cap \gamma_n H \gamma_n^{-1}$ is infinite (see [22]). If such a supremum is infinite, we say that H is of *infinite height* in Γ. Otherwise, H is said to be of *finite height* in Γ. A quasiconvex subgroup of a hyperbolic group is of finite height, by a result of [22].

Definition IV.2.5 (Finite height property)**.** We say that the $E\mathcal{Z}$-complex of classifying spaces $EG(\mathcal{Y})$ compatible with the complex of groups $G(\mathcal{Y})$ satisfies the *finite height property* if for every pair of simplices $\sigma \subset \sigma'$ of Y, $G_{\sigma'}$ is of finite height in G_σ.

IV.3 The geometry of the action.

In this section, we gather a few geometric tools that will be used to construct a topology on $\overline{EG} = EG \cup \partial G$. From now on, we assume that the $E\mathcal{Z}$-complex of classifying spaces $EG(\mathcal{Y})$ compatible with $G(\mathcal{Y})$ satisfies the limit set property IV.2.4 and the finite height property IV.2.5. We further assume that the action of G on X is acylindrical and we fix an *acylindricity constant* $A > 0$, that is, a constant such that every subcomplex of X of diameter at least A has a finite pointwise stabiliser.

IV.3.1 Domains and their geometry.

In this section, we study the topological properties of the identifications made to build the boundary of G.

Definition IV.3.1. Let $\xi \in \partial_{Stab} G$. We define $D(\xi)$, called the *domain* of ξ, as the subcomplex of X whch is the union of the simplices σ such that $\xi \in \partial G_\sigma$. We denote by $V(\xi)$ the set of vertices of $D(\xi)$.

The aim of this paragraph is to prove the following:

Proposition IV.3.2. *Domains are finite convex subcomplexes of X with a uniformly bounded number of simplices.*

The containment lemma IV.1.3 and Proposition IV.3.2 imply the following:

Corollary IV.3.3. *For every $\xi \in \partial_{Stab}G$, there exists an integer d_ξ such that $D(\xi)$ has at most d_ξ simplices, and such that a geodesic segment in the open simplicial neighbourhood of $D(\xi)$ meets at most d_ξ open simplices. Furthermore, there exists an upper bound $d_{\max}$ on the set of integers $d_\xi, \xi \in \partial_{Stab}G$.*

Recall that $\Omega(\mathcal{Y})$ is defined in IV.2.2 as the disjoint union of the ∂G_v's ($v \in V(X)$) and that $\partial_{Stab}G$ is a quotient of $\Omega(\mathcal{Y})$ defined by making identifications along edges of X. We start by proving the following proposition:

Proposition IV.3.4. *Let v be a vertex of X. Then the projection $\pi : \partial G_v \to \partial G$ is injective.*

Definition IV.3.5. Let $\xi \in \partial_{Stab}G$. A *ξ-path* is the data $\{(v_i)_{0\leq i\leq n}, (\xi_i)_{0\leq i\leq n}, (x_i)_{1\leq i\leq n}\}$ of:

- a sequence $v_0, \dots, v_n$ of adjacent vertices of X,
- a sequence $\xi_0, \dots, \xi_n$ of elements of $\Omega(\mathcal{Y})$, such that $\xi_i \in \partial G_{v_i}$ for every i, and such that each ξ_i is in the equivalence class ξ,
- a sequence $x_1, \dots, x_n$ of elements of $\Omega(\mathcal{Y})$, such that $x_i \in \partial G_{[v_{i-1}, v_i]}$ for every i, and such that $\phi_{v_{i-1},[v_{i-1},v_i]}(x_i) = \xi_{i-1}$ (resp $\phi_{v_i,[v_{i-1},v_i]}(x_i) = \xi_i$).

To lighten notations, a ξ-path will sometimes just be denoted $[v_0, \dots, v_n]_\xi$. The path in the 1-skeleton of X induced by a ξ-path is called the *support* of $[v_0, \dots, v_n]_\xi$, and denoted $[v_0, \dots, v_n]$. If $v_0 = v_n$, a ξ-path will rather be called a *ξ-loop*.

Lemma IV.3.6. *Let $v_0, \dots, v_n$ be vertices of X, $H = \cap_{0\leq i\leq n} G_{v_i}$, and K be a connected subcomplex of X pointwise fixed by H. Suppose that H is infinite, and let $\xi \in \partial_{Stab}G$ such that, in G_{v_0}, we have*

$$\xi \in \Lambda H \subset \partial G_{v_0}.$$

Then $\xi \in \Lambda H \subset \partial G_\sigma$ for every simplex σ of K, hence $K \subset D(\xi)$.

Proof. As K is connected, it is enough to prove that for every path of simplices $\sigma_0 = v_0, \dots, \sigma_d$ contained in K, we have $\xi \in \partial H \subset \partial G_{\sigma_d}$. Now this follows from an easy induction on the number of simplices contained in such a path. □

Lemma IV.3.7. *Let $\xi \in \partial_{Stab}G$, $[v_0, \dots, v_n]_\xi$ a ξ-path and $H = \cap_{0\leq i\leq n} G_{v_i}$. Then*

- *H is infinite,*
- *$\xi \in \Lambda H \subset \partial G_{v_i}$ for every $i = 0, \dots, n$.*

Proof. We show the result by induction on $n \geq 1$. The result is immediate for $n = 1$ by definition of $\sim$. Suppose the result true up to rank n and let $\xi \in \partial_{Stab}G$ together with a ξ-path $[v_0, \ldots, v_{n+1}]_\xi$. By restriction, we get a ξ-path $[v_0 \ldots, v_n]_\xi$ for which the result is true by the induction hypothesis. Thus $\xi \in \Lambda(\cap_{0 \leq i \leq n} G_{v_i}) \subset \partial G_{v_n}$. But since ξ is also in $\partial G_{[v_n, v_{n+1}]} = \Lambda G_{[v_n, v_{n+1}]}$ by assumption, we get

$$\xi \in \Lambda\Big(\bigcap_{0 \leq i \leq n} G_{v_i}\Big) \cap \Lambda G_{[v_n, v_{n+1}]} = \Lambda\Big(\bigcap_{0 \leq i \leq n+1} G_{v_i}\Big) \subset \partial G_{v_n},$$

the previous equality following from the limit set property IV.2.4. Now, by Lemma IV.3.6, we get $\xi \in \Lambda(\cap_{0 \leq i \leq n+1} G_{v_i}) \subset \partial G_{v_i}$ for every $i = 0, \ldots, n+1$, which concludes the induction. □

Proof of Proposition IV.3.4. Let ξ, ξ' be two elements of $\Omega(\mathcal{Y})$ in the image of ∂G_v, that are equivalent for the equivalence relation $\sim$. Then there exists a ξ-loop $\{(v_i)_{0 \leq i \leq n}, (\xi_i)_{0 \leq i \leq n}, (x_i)_{1 \leq i \leq n}\}$ with $\xi_0 = \xi$, $\xi_n = \xi'$. It is enough to prove the result when the support $[v_0, \ldots, v_n]$ of that ξ-loop is injective. Let Y be the set of all points on a geodesic between two points of $[v_0, \ldots, v_n]$. By the previous lemma, there is an infinite subgroup H of G stabilising pointwise $v_0, \ldots, v_n$. As X is CAT(0), H also stabilises pointwise every point of Y. As $[v_0, \ldots, v_n]$ is contractible inside Y, the finiteness lemma IV.1.5 implies that we can choose a finite 2-complex F such that the loop $[v_0, \ldots, v_n]$ is contractible inside F, and such that F is pointwise fixed by H. We call such a subcomplex a *hull* of the loop $[v_0, \ldots, v_n]$. Hence the result will follow from the following fact, which we now prove by induction.

(H_d): For every $\xi \in \partial_{Stab}G$ and every ξ-loop $\{(v_i)_{0 \leq i \leq n}, (\xi_i)_{0 \leq i \leq n}, (x_i)_{1 \leq i \leq n}\}$ admitting a hull containing at most d triangles, we have $\xi_0 = \xi_n$.

If $d = 1$, then $n = 2$, and the hull considered is just a single triangle σ. Since $H \subset G_\sigma$ because H stabilises σ pointwise, we can choose $x \in \partial G_\sigma$ such that $\phi_{v_1,\sigma}(x) = \xi_1$. From the commutativity of the diagram of embeddings for a simplex, it follows that $\phi_{[v_0,v_1],\sigma}(x) = x_1$ and $\phi_{[v_1,v_2],\sigma}(x) = x_2$. Hence $\xi_0 = \phi_{v_0,[v_0,v_1]}(x_1) = \phi_{v_0,\sigma}(x) = \phi_{v_0,[v_2,v_0]}(x_2) = \xi_2$.

Suppose the result true up to rank d, and let $\xi \in \partial_{Stab}G$, together with a ξ-loop $\{(v_i)_{0 \leq i \leq n}, (\xi_i)_{0 \leq i \leq n}, (x_i)_{1 \leq i \leq n}\}$ admitting a hull F containing at most $d+1$ triangles. Choose any triangle σ of F containing the segment $[v_1, v_2]$. As σ is stabilised by H, we can find $x \in \partial G_\sigma$ such that $\phi_{v_1,\sigma}(x) = \xi_1$. There are now two possible cases, depending of the nature of σ:

- If another side of σ is contained in the support of the ξ-loop, for example $[v_2, v_3]$, we set $x' = \phi_{[v_1,v_3],\sigma}(x)$.

  Now the commutativity of the diagram of embeddings for σ yields the following new ξ-loop

$$\{(v_0, v_1, v_3, v_4, \ldots, v_n), (\xi_0, \xi_1, \xi_3, \ldots, \xi_n), (x_1, x', x_4, \ldots, x_n)\}.$$

A hull for that new loop is given by the closure of $F \setminus \sigma$, thus containing at most d triangles, and we are done by induction.

- If no other side of σ is contained in the support of the ξ-loop, we set a to be the remaining vertex of σ, $\alpha = \phi_{a,\sigma}(x)$, $x_2 = \phi_{[v_1,a],\sigma}(x)$ and $x_2' = \phi_{[a,v_2],\sigma}(x)$.

  The commutativity of the diagram of embeddings for σ yields the following new ξ-loop:

$$\{(v_0, v_1, a, v_2, \ldots, v_n), (\xi_0, \xi_1, \alpha, \xi_2, \ldots, \xi_n), (x_1, x_2, x_2', x_3, \ldots, x_n)\}.$$

  A hull for that new loop is given by the closure of $F \setminus \sigma$, thus containing at most d triangles, and we are done by induction. □

Proof of Proposition IV.3.2. Convexity: Let x, x' be two points of $D(\xi)$. Let v (resp. v') be a vertex of σ_x (resp. $\sigma_{x'}$). We can thus find a ξ-path $\{(v_i)_{0\leq i\leq n}, (\xi_i)_{0\leq i\leq n}, (x_i)_{1\leq i\leq n}\}$ with $v_0 = v$ and $v_n = v'$. As $\xi \in \partial G_{\sigma_x}$ and $\xi \in \partial G_{\sigma_{x'}}$, we can assume without loss of generality that its support $[v_0, \ldots, v_n]$ contains all the vertices of σ_x and $\sigma_{x'}$. By Lemma IV.3.7, this implies that the subgroup $H = \cap_{0\leq i\leq p} G_{v_i}$ is infinite and that $\xi \in \Lambda H \subset \partial G_{v_0}$. Now since H fixes pointwise all the vertices of σ_x and $\sigma_{x'}$, and since X is CAT(0), H also fixes pointwise the geodesic segment $[x, x']$. But by Lemma IV.3.6, the fixed-point set of H is contained in $D(\xi)$, hence so is $[x, x']$. Thus $D(\xi)$ is convex.

Finiteness: Let σ be a simplex of $D(\xi)$ and $\sigma_1, \sigma_2, \ldots$ be a (possibly empty) sequence of simplices containing strictly σ and contained in $D(\xi)$. It follows from the proof of Proposition IV.3.4 that $\xi \in \partial G_{\sigma_i} \subset \partial G_\sigma$ for every i. Thus, the limit set property IV.2.4, the finite height property IV.2.5 and the cocompactness of the action imply that there can be only finitely many such simplices. Thus $D(\xi)$ locally finite. To prove that it is also bounded, consider x, x' two points of $D(\xi)$. By Lemma IV.3.7 the stabiliser of $\{x, x'\}$ is infinite. Thus the domain of ξ has a diameter bounded above by the acylindricity constant. The complex $D(\xi)$ is locally finite and bounded, hence finite. Moreover, it is clear from the above argument that the bound can be chosen uniform on ξ. □

IV.3.2 Nestings and Families.

Definition IV.3.8 (the convergence property)**.** We say that an $E\mathcal{Z}$-complex of classifying spaces compatible with $G(\mathcal{Y})$ satisfies the *convergence property* if, for every pair of simplices $\sigma \subset \sigma'$ in Y and every injective sequence $(g_n G_{\sigma'})$ of cosets of $G_\sigma / G_{\sigma'}$, there exists a subsequence such that $(g_{\varphi(n)} \overline{EG_{\sigma'}})$ uniformly converges to a point in $\overline{EG_\sigma}$.

From now on, besides the limit set property IV.2.4, the finite height property IV.2.5 and the acylindricity assumption, we assume that the $E\mathcal{Z}$-complex of classifying spaces $EG(\mathcal{Y})$ satisfies the convergence property IV.3.8.

Definition IV.3.9. Let $\xi \in \partial_{Stab} G$, v a vertex of $D(\xi)$, and U a neighbourhood of ξ in $\overline{EG_v}$. We say that a subneighbourhood $V \subset U$ containing ξ is *nested* in U if its closure is contained in U and for every simplex σ of $\mathrm{st}(v)$ not contained in $D(\xi)$, we have

$$\overline{EG_\sigma} \cap V \neq \varnothing \Rightarrow \overline{EG_\sigma} \subset U.$$

Lemma IV.3.10 (nesting lemma)**.** *Let $\xi \in \partial_{Stab} G$, v a vertex of $D(\xi)$ and U a neighbourhood of ξ in $\overline{EG_v}$. Then there exists a subneighbourhood of ξ in $\overline{EG_v}$, $V \subset U$, which is nested in U.*

Proof. We show this by contradiction. Consider a countable basis $(V_n)_n$ of neighbourhoods of ξ in $\overline{EG_v}$, and suppose that for every n, there exists a simplex $\sigma_n \in \mathrm{st}(v) \setminus D(\xi)$ such that $\overline{EG_{\sigma_n}} \cap V_n \neq \varnothing$ and $\overline{EG_{\sigma_n}} \subsetneq U$. Up to a subsequence, we can assume that $(\sigma_n)_n$ is injective. By cocompactness of the action, we can also assume that all the σ_n cover a unique simplex $\overline{\sigma}$ of Y. Now the convergence property IV.3.8 implies that there should exist a subsequence $\sigma_{\lambda(n)}$ such that $\overline{EG_{\sigma_{\lambda(n)}}}$ uniformly converges to a point in $\overline{EG_v}$, a contradiction. □

Since, in ∂G, boundaries of stabilisers of vertices are glued together along boundaries of stabilisers of edges, we will construct neighbourhoods in $\overline{EG}$ of a point $\xi \in \partial_{Stab} G$ using neighbourhoods of the representatives of ξ in the various $\overline{EG_v}$, where v runs over the vertices of the domain of ξ.

Definition IV.3.11 (ξ-family)**.** Let $\xi \in \partial_{Stab} G$. A collection $\mathcal{U}$ of open sets $\{U_v, v \in V(\xi)\}$ is called a *ξ-family* if for every pair of vertices v, v' of X that are joined by an edge e and every $x \in \overline{EG_e}$,

$$\phi_{v,e}(x) \in U_v \Leftrightarrow \phi_{v',e}(x) \in U_{v'}.$$

Proposition IV.3.12. *Let $\xi \in \partial_{Stab} G$. For every vertex v of $D(\xi)$, let U_v be a neighbourhood of ξ in $\overline{EG_v}$. Then there exists a ξ-family $\mathcal{U}'$ such that $U'_v \subset U_v$ for every vertex v of $D(\xi)$.*

Proof. For every simplex σ of $D(\xi)$, we construct open sets U'_σ by induction on $\dim(\sigma)$, starting with simplices of maximal dimension, that we denote d.

If $\dim(\sigma) = d$, we set

$$U'_\sigma = \bigcap_{v \in \sigma} \phi_{v,\sigma}^{-1}(U_v).$$

Assume the U'_σ constructed for simplices of dimension at least $k \leq d$, and let σ_0 be of dimension $k-1$. If no simplex of dimension $\geq k$ contains σ_0, set

$$U'_{\sigma_0} = \bigcap_{v \in \sigma} \phi_{v,\sigma_0}^{-1}(U_v).$$

Otherwise, since the $\phi_{\sigma,\sigma'}$ are embeddings,

$$\bigcup_{\substack{\sigma_0\subset\sigma\subset D(\xi)\\ \dim(\sigma)=k}} \phi_{\sigma_0,\sigma}(U'_\sigma)$$

is open in

$$\bigcup_{\substack{\dim(\sigma)=k\\ \sigma_0\subset\sigma\subset D(\xi)}} \phi_{\sigma_0,\sigma}(\overline{EG_\sigma}).$$

We can thus write it as the trace of an open set U'_{σ_0} of $\overline{EG_{\sigma_0}}$. This yields for every vertex v of $D(\xi)$ a new open set U'_v. By intersecting it with U_v, we can further assume that $U'_v \subset U_v$. This new collection of neighbourhoods clearly satisfies the desired property. □

Definition IV.3.13. Let $\xi \in \partial_{Stab}G$, together with two ξ-families $\mathcal{U},\mathcal{U}'$. We say that $\mathcal{U}'$ is *nested* in $\mathcal{U}$ if for every vertex v of $D(\xi)$, U'_v is nested in U_v. Furthermore we say that $\mathcal{U}'$ is n-nested in $\mathcal{U}$ if there exist ξ-families

$$\mathcal{U}' = \mathcal{U}^{[0]} \subset \ldots \subset \mathcal{U}^{[n]} = \mathcal{U}$$

with $\mathcal{U}^{[i]}$ nested in $\mathcal{U}^{[i+1]}$ for every $i = 0, \ldots, n-1$.

IV.4 A geometric toolbox.

We now prove some results which will be our main tools in studying $\overline{EG}$ and ∂G. Since the proofs in this section rely heavily on the geometry of X, we start with a few definitions.

Definition IV.4.1. Let $\xi \in \partial_{Stab}G$, $x \in X$, $\eta \in \partial X$ and $\varepsilon \in (0,1)$.

Let d be the simplicial metric on X, and choose a basepoint $v_0 \in X$. We denote by $[v_0, x]$ the unique geodesic segment from v_0 to x, and by $\gamma_x : [0, d(v_0, x)] \to X$ its parametrisation. We denote by $[v_0, \eta)$ the unique geodesic ray from v_0 to η, and by $\gamma_\eta : [0, \infty) \to X$ its parametrisation.

We denote by $D^\varepsilon(\xi)$ the open ε-neighbourhood of $D(\xi)$.

We say that a geodesic in X parametrised by γ *goes through* (resp. *enters*) $D^\varepsilon(\xi)$ if there exist t_0 such that $\gamma(t_0) \in D^\varepsilon(\xi)$ and $t_1 > t_0$ such that $\gamma(t_1) \notin D^\varepsilon(\xi)$ (resp. if there exists t_0 such that $\gamma(t_0) \in D^\varepsilon(\xi)$) .

If the geodesic $[v_0, x]$ goes through $D^\varepsilon(\xi)$, we define an *exit simplex* $\sigma_{\xi,\varepsilon}(x)$ as the first simplex touched by $[v_0, x]$ after leaving $D^\varepsilon(\xi)$. If $x \in D^\varepsilon(\xi)$, we set $\sigma_{\xi,\varepsilon}(x) = \sigma_x$.

Note that, by the assumption on the distance from a simplex to the boundary of its closed simplicial neighbourhood, we always have $D^\varepsilon(\xi) \subset \mathrm{N}(D(\xi))$.

Definition IV.4.2. Let $\xi \in \partial_{Stab}G$, $\mathcal{U}$ a ξ-family and $\varepsilon \in (0,1)$. We define $\mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$ (resp. $\widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$) as the set of points x of $X \setminus D(\xi)$ such that the geodesic $[v_0, x]$ goes through (resp. enters) $D^\varepsilon(\xi)$ and such that for some vertex v of $D(\xi)$ (hence for every by Definition IV.3.11) contained in the exit simplex $\sigma_{\xi,\varepsilon}(x)$, we have, in $\overline{EG_v}$:

$$\overline{EG}_{\sigma_{\xi,\varepsilon}(x)} \subset U_v.$$

Definition IV.4.3. For $\xi \in \partial_{Stab}G$ and $\mathcal{U}$ a ξ-family (Definition IV.3.11), we define the subcomplex $\mathrm{N}_{\mathcal{U}}(D(\xi))$ as the union of closed simplices $\sigma \subset \overline{\mathrm{N}}(D(\xi))$ such that for some (hence for every) vertex v of $D(\xi) \cap \sigma$, we have, in $\overline{EG_v}$:

$$\overline{EG_\sigma} \cap U_v \neq \varnothing.$$

IV.4.1 The crossing lemma.

Lemma IV.4.4 (crossing lemma)**.** *Let $\xi \in \partial_{Stab}G$, $\mathcal{U}$, $\mathcal{U}'$ two ξ-families, and $\sigma_1, \ldots, \sigma_n$ ($n \geq 1$) a path of open simplices contained in $N\big(D(\xi)\big) \setminus D(\xi)$. Suppose that $\mathcal{U}'$ is n-nested in $\mathcal{U}$ (Definition IV.3.13), and that $\sigma_1 \subset N_{\mathcal{U}'}\big(D(\xi)\big)$. Then for every $k \in \{1, \ldots, n\}$ and every vertex v of $D(\xi)$ contained in σ_k, we have $\overline{EG_{\sigma_k}} \subset U_v$ in $\overline{EG_v}$.*

Proof. We prove the result by induction on n, by using the definition of nested families.

The result for $n = 1$ follows from the definition of a nested family. Suppose the result true for $1, \ldots, n$, and let $\sigma_1, \ldots, \sigma_{n+1}$ be a path of simplices in $\mathrm{N}\big(D(\xi)\big) \setminus D(\xi)$ and $\mathcal{U}^{[0]} \subset \ldots \subset \mathcal{U}^{[n+1]} = \mathcal{U}$. By induction, the result is true for the path $\sigma_1, \ldots, \sigma_n$ and the filtration $\mathcal{U}^{[0]} \subset \ldots \subset \mathcal{U}^{[n]}$, so the only inclusion to be proved is the aforementioned one for σ_{n+1}.

If $\sigma_n \subset \sigma_{n+1}$, every vertex v of σ_n is also a vertex of σ_{n+1}, so the result is already true for vertices of $D(\xi)$ contained in σ_n. Now by the definition of ξ-families (see Definition IV.3.11), this implies the result for every vertex of $D(\xi) \cap \sigma_{n+1}$.

Suppose now that $\sigma_n \supset \sigma_{n+1}$, and let v be a vertex of $D(\xi)$ contained in σ_{n+1}. Since v is also in σ_n, $\overline{EG_{\sigma_n}} \subset U^{[n]}_{v_d}$ in $\overline{EG_{\sigma_n}}$, so we have $\overline{EG}_{\sigma_{n+1}} \cap U^{[n]}_{v_n} \neq \varnothing$, which in turn implies $\overline{EG}_{\sigma_{n+1}} \subset U^{[n+1]}_v$ since $\mathcal{U}^{[n]}$ is nested in $\mathcal{U}^{[n+1]}$. Now by the definition of ξ-families IV.3.11, the same result holds for every vertex v of $D(\xi)$ contained in σ_{n+1}.

□

IV.4.2 The geodesic reattachment lemma.

Recall that Definition IV.3.3 yields for every $\xi \in \partial_{Stab}G$ a constant $d_\xi \leq d_{\max}$ such that $D(\xi)$ contains at most d_ξ simplices and such that a geodesic contained in the open simplicial neighbourhood of $D(\xi)$ meets at most $d_{\max}$ open simplices.

Definition IV.4.5 (refined families)**.** Let $n \geq 1$. By Lemma IV.1.7, we can choose a constant m such that the following holds:

Let K be a convex subcomplex of X and K' a connected subcomplex of X, both containing at most $\max(n, d_{\max})$ simplices. Let $x, y \in K$ and $x', y' \in K'$ and assume that there exists a path in K' between x and y that does not meet K. Let τ, τ' be two simplices of $\mathrm{N}(K) \setminus K$ such that the geodesic segment $[x, x']$ (resp. $[y, y']$) meets the interior of τ (resp. τ'). Then there exists a path of simplices in $\mathrm{N}(K) \setminus K$ of length at most m between τ and τ'.

Let $\xi \in \partial_{Stab} G$, $\mathcal{U}$ a ξ-family. A ξ-family that is m-nested in $\mathcal{U}$ is said to be *n-refined* in $\mathcal{U}$. For n the number of simplices of $D(\xi)$, we denote by m_ξ such a choice of m.

Lemma IV.4.6. *Let $\xi \in \partial_{Stab} G$. There exists a ξ-family $\mathcal{V}_\xi$ such that for every vertex v of $D(\xi)$ and every simplex σ of $\big(st(v) \setminus D(\xi)\big) \cap Geod\big(v_0, D(\xi)\big)$, we have $(V_\xi)_v \cap \overline{EG_\sigma} = \varnothing$.*

Proof. Let σ a simplex of $\mathrm{N}(D(\xi)) \setminus D(\xi)$ whose interior meets $\mathrm{Geod}(v_0, D(\xi))$. Let v be a vertex of $D(\xi) \cap \sigma$. Let U_v be a neighbourhood of ξ in $\overline{EG_v}$ that is disjoint from $\overline{EG_\sigma}$. For every other vertex w of $D(\xi)$, set $U_w = \overline{EG_w}$. By Proposition IV.3.12, we choose a ξ-family $\mathcal{V}_\xi$ that is $(d_\xi + 1)$-refined in the collection of open sets $\{U_w, w \in V(\xi)\}$. The result now follows from Definition IV.4.5. □

Lemma IV.4.7. *Let $\xi \in \partial_{Stab} G$. Let $\mathcal{U}$ be a ξ-family that is m_ξ-nested in $\mathcal{V}_\xi$ (recall that $\mathcal{V}_\xi$ is assumed to satisfy Lemma IV.4.6). Let $x \in X \setminus D(\xi)$ be such that there exists a simplex $\sigma \subset \Big(N(D(\xi)) \setminus D(\xi)\Big)$ that meets $Geod\big(x, D(\xi)\big)$ and such that for some (hence any) vertex v of $\sigma \cap D(\xi)$ we have $\overline{EG_\sigma} \subset U_v$. Then $x \notin Geod(v_0, D(\xi))$.*

Proof. We prove the lemma by contradiction. Let x and σ be as in the statement of the lemma. Let $z \in D(\xi)$ be such that $x \in [v_0, z]$ and $z' \in D(\xi)$ be such that the geodesic segment $[x, z']$ meets σ. Let σ' be the last simplex touched by $[v_0, z]$ before meeting $D(\xi)$, and v' a vertex of σ'.

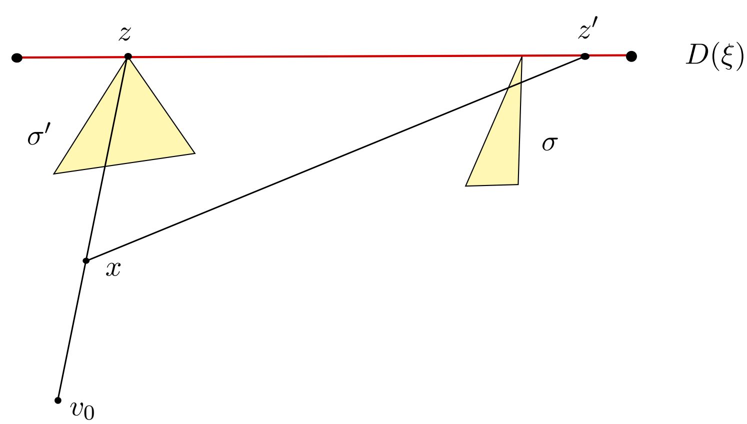

Figure IV.2.

Since $\mathcal{U}$ is m_ξ-nested in $\mathcal{V}_\xi$, it follows from the inclusion $\overline{EG_\sigma} \subset U_v$ and Lemma IV.1.7 that $\overline{EG_{\sigma'}} \subset (V_\xi)_{v'}$, contradicting the definition of $\mathcal{V}_\xi$.

□

The next lemma gives a useful criterion that ensures that a given path is a global geodesic.

Lemma IV.4.8 (geodesic reattachment lemma)**.** *Let $\xi \in \partial_{Stab} G$, $\mathcal{V}$ a ξ-family satisfying Lemma IV.4.6, $\mathcal{U}$ a ξ-family which is $(m_\xi + d_\xi)$-nested in $\mathcal{V}$, and $x \in X \setminus D(\xi)$. Suppose that there exists a simplex $\sigma \subset N(D(\xi)) \setminus D(\xi)$ that meets $Geod\big(x, D(\xi)\big)$ such that for some (hence any) vertex v of $\sigma \cap D(\xi)$ we have $\overline{EG_\sigma} \subset U_v$. Then $[v_0, x]$ meets $D(\xi)$ and $x \in \widetilde{Cone}_{\mathcal{V},\varepsilon}(\xi)$ for every $\varepsilon \in (0, 1)$.*

In such a case, the geodesic from v_0 to x meets $D(\xi)$, and is the concatenation of a geodesic segment in $\mathrm{Geod}(v_0, D(\xi))$ and a geodesic in $\mathrm{Geod}(D(\xi), x)$.

Proof. Let $K = \mathrm{Geod}(v_0, D(\xi)) \cup \mathrm{Geod}(D(\xi), x)$ and let $[v_0, x]_K$ be the geodesic from v_0 to x in K (which meets finitely many simplices by Lemma IV.1.5). Our aim is to prove that $[v_0, x]_K = [v_0, x]$. By Lemma IV.4.7, $x \notin \mathrm{Geod}(v_0, D(\xi))$. As $D(\xi)$ is convex by Proposition IV.3.2, let $v_1, v_2 \in D(\xi)$ be such that $[v_0, x]_K = [v_0, v_1] \cup [v_1, v_2] \cup [v_2, x]$ and such that $[v_0, v_1)$ and $(v_2, x]$ do not meet $D(\xi)$. Let $\varepsilon \in (0, 1)$. Let $a \in [v_0, v_1]$ be such that $d(a, v_1) = \varepsilon$. If $x \notin D^\varepsilon(\xi)$ let $b \in [v_2, x]$ be such that $d(v_2, b) = \varepsilon$. Otherwise, let $b = x$. Since X is CAT(0), it is enough to prove that $[v_0, x]_K$ is a local geodesic at every point. We already have that $[v_0, v_1] \cup [v_1, v_2]$ and $[v_1, v_2] \cup [v_2, x]$ are geodesics, so it is sufficient to prove the result when $v_1 = v_2$. We thus have

$$[v_0, x]_K = [v_0, v_1] \cup [v_1, x],$$

with $[v_0, v_1] \subset \mathrm{Geod}(v_0, D(\xi))$ and $[v_1, x] \subset \mathrm{Geod}(D(\xi), x)$. Assume by contradiction that $[v_0, x]_K$ is not a local geodesic at v_1. Then the geodesic segment $[a, b]$ does not meet $D(\xi)$. This geodesic segment yields a path of simplices between σ_a and σ_b of length at most d_ξ in $\mathrm{N}(D(\xi)) \setminus D(\xi)$. Furthermore, there is a path of simplices between σ and σ_b of length at most m_ξ in $\mathrm{N}(D(\xi)) \setminus D(\xi)$ by Definition IV.4.5. Thus, there is a path of simplices between σ and σ_a of length at most $m_\xi + d_\xi$ in $\mathrm{N}(D(\xi)) \setminus D(\xi)$. But since $\overline{EG_b} \subset U_v$ and $\mathcal{U}$ is $(m_\xi + d_\xi)$-nested in $\mathcal{V}$, the crossing lemma IV.4.4 implies $\overline{EG_a} \subset V_v$, which contradicts the fact that $\mathcal{V}$ satisfies Lemma IV.4.6.

Thus $[v_0, x]_K = [v_0, x]$ and $\sigma_b = \sigma_{\xi,\varepsilon}(x)$. It follows from the above discussion that for some (hence every) vertex v' of $\sigma_{\xi,\varepsilon}(x)$ we have $\overline{EG}_{\sigma_{\xi,\varepsilon}(x)} \subset V_{v'}$, hence $x \in \widetilde{\mathrm{Cone}}_{\mathcal{V},\varepsilon}(\xi)$. □

From now on, every ξ-family will be assumed to be contained in a ξ-family $\mathcal{U}_\xi$ satisfying Lemma IV.4.8.

As a consequence, we get the following:

Corollary IV.4.9. *Let $\xi \in \partial_{Stab} G$, $\mathcal{U}$ a ξ-family and $\varepsilon \in (0,1)$. Then for every $x \in \widetilde{Cone}_{\mathcal{U},\varepsilon}(\xi)$, the geodesic segment $[v_0, x]$ meets $D(\xi)$.*

Proof. By Lemma IV.4.7 applied to x and $\sigma_{\xi,\varepsilon}(x)$, we get $x \notin \mathrm{Geod}(v_0, D(\xi))$. Let y be a point of $\sigma_{\xi,\varepsilon}(x) \cap [v_0, x] \cap D^\varepsilon(\xi)$. It follows from the geodesic reattachment lemma IV.4.8 applied to y and $\sigma_{\xi,\varepsilon}(x)$ that $[v_0, y]$, hence $[v_0, x]$, meets $D(\xi)$. □

IV.4.3 The refinement lemma.

Lemma IV.4.10 (refinement lemma)**.** *Let $\xi \in \partial_{Stab} G$, $\mathcal{U}$ a ξ-family and $n \geq 1$. Let $\mathcal{U}'$ be a ξ-family which is n-refined in $\mathcal{U}$. Then the following holds:*

For every $\varepsilon \in (0,1)$ and every path of simplices $\sigma_1, \ldots, \sigma_n$ in $X \setminus D(\xi)$ such that there exists a point $x_1 \in \sigma_1$ such that $[v_0, x_1]$ enters $D^\varepsilon(\xi)$ and $\sigma_{\xi,\varepsilon}(x_1) \subset \mathrm{N}_{\mathcal{U}'}\big(D(\xi)\big)$, we have

$$\sigma_1, \ldots, \sigma_n \subset \widetilde{Cone}_{\mathcal{U},\varepsilon}(\xi).$$

Proof. Let us prove that for every $x \in \cup_{1 \leq i \leq n} \sigma_i$, the geodesic segment $[v_0, x]$ meets $D(\xi)$. Let $x_1 \in \sigma_1$ such that $\sigma_{\xi,\varepsilon}(x_1) \subset \mathrm{N}_{\mathcal{U}'}(D(\xi))$. Note that Corollary IV.4.9 implies that $[v_0, x_1]$ meets $D(\xi)$. Let v be a vertex of $D(\xi) \cap \sigma_{\xi,\varepsilon}(x_1)$.

Let $x \in \cup_{1 \leq i \leq n} \sigma_i$ and σ be a simplex of $\mathrm{N}(D(\xi)) \setminus D(\xi)$ touched by $[v, x]$ after leaving $D(\xi)$. Let also w be a vertex of $\sigma \cap D(\xi)$. We can apply Lemma IV.1.7 to the geodesic segments $[v, x]$ and (a portion of) $[v_0, x_1]$, and to simplices σ and $\sigma_{\xi,\varepsilon}(x_1)$. Since $\overline{EG}_{\sigma_{\xi,\varepsilon}(x_1)} \subset U'_v$ and $\mathcal{U}'$ is n-refined in $\mathcal{U}$, we get $\overline{EG_\sigma} \subset U_w$. Thus the geodesic reattachment lemma IV.4.8 implies that $[v_0, x]$ meets $D(\xi)$.

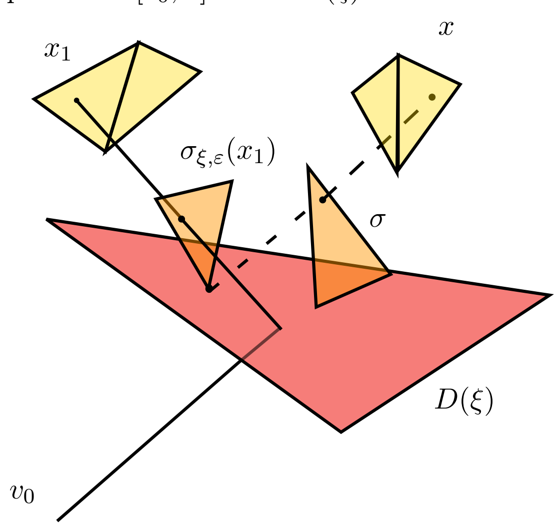

Figure IV.3.

Let $x \in \cup_{1\leq i\leq n}\sigma_i$ and let w be a vertex of $\sigma_{\xi,\varepsilon}(x) \cap D(\xi)$. We apply apply once again Lemma IV.1.7, this time to portions of the geodesic segments $[v_0, x]$ and $[v_0, x_1]$, and to simplices $\sigma_{\xi,\varepsilon}(x)$ and $\sigma_{\xi,\varepsilon}(x_1)$. Now since $\mathcal{U}'$ is n-refined in $\mathcal{U}$ and $\overline{EG}_{\sigma_{\xi,\varepsilon}(x_1)} \subset U'_v$, we get $\overline{EG}_{\sigma_{\xi,\varepsilon}(x)} \subset U_w$, hence $x \in \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$. □

IV.4.4 The star lemma.

Lemma IV.4.11 (star lemma). *Let $\xi \in \partial_{Stab}G$, $\varepsilon \in (0,1)$ and $x \in X \setminus D^\varepsilon(\xi)$ such that the geodesic segment $[v_0, x]$ goes through $D^\varepsilon(\xi)$. Then there exists $\delta > 0$ such that for every $y \in B(x,\delta) \setminus D^\varepsilon(\xi)$, the geodesic segment $[v_0, y]$ goes through $D^\varepsilon(\xi)$. Furthermore, for every $y \in B(x,\delta) \setminus D^\varepsilon(\xi)$, we have*

$$\sigma_{\xi,\varepsilon}(y) \subset st(\sigma_{\xi,\varepsilon}(x)).$$

Proof. Let $T = \mathrm{dist}(v_0, x)$, and let $\gamma_x : [0,T] \to X$ be the parametrisation of the geodesic segment $[v_0, x]$. Let $t_0 > 0$ such that $[v_0, x]$ leaves $D^\varepsilon(\xi)$ at time t_0. Since $D(\xi)$ is convex by Proposition IV.3.2, the map $z \mapsto \mathrm{dist}(z, D(\xi))$ is convex. Thus, there exists $r > 0$ such that

$$\gamma_x\big([t_0 - r, t_0)\big) \subset D^\varepsilon(\xi),$$
$$\gamma_x\big(\,[t_0 - r, t_0]\,\big) \subset \mathrm{st}\left(\sigma_{\xi,\varepsilon}(x)\right).$$

We also choose $\tau > 0$ such that for every y_-, y_+ in the τ-neighbourhood of $\gamma_x\big(\,[t_0 - r, t_0]\,\big)$, the geodesic segment $[y_-, y_+]$ is contained in $\mathrm{st}(\sigma_{\xi,\varepsilon}(x))$.
Let

$$k = \varepsilon - \mathrm{dist}(\gamma_x(t_0 - r), D(\xi)) > 0.$$

We set $\delta_1 = 1/10 \cdot \min(k, \tau, r)$. If $x \in \overline{D^\varepsilon(\xi)}$, set $\delta = \delta_1$. If $x \notin \overline{D^\varepsilon(\xi)}$, we can assume without loss of generality that $\delta_1 < 1/10 \cdot (T - t_0)$. By convexity of the distance, we have $d\big(\gamma_x(t_0 + \delta_1), D(\xi)\big) > \varepsilon$, and we set $\delta = 1/2 \cdot \min\big(\delta_1, d(\gamma_x(t_0 + \delta_1), D^\varepsilon(\xi))\big) > 0$.

Let $y \in B(x,\delta) \setminus D^\varepsilon(\xi)$, and let γ_y be its parametrisation.
Since $\delta \leq r$, we have $d(v_0, y) \geq t_0 - r$. Now, γ_x and γ_y parametrise geodesics starting at v_0 and such that $d(x,y) < \delta$, so since X is a CAT(0)-space, we get $d(\gamma_x(t_0 - r), \gamma_y(t_0 - r)) \leq 2\delta \leq \tau$. The inequality $10\delta \leq k$ now implies

$$\begin{aligned} d\Big(\gamma_y(t_0 - r), D(\xi)\Big) &\leq d\Big(\gamma_x(t_0 - r), D(\xi)\Big) + d\Big(\gamma_x(t_0 - r), \gamma_y(t_0 - r)\Big) \\ &\leq (\varepsilon - 10\delta) + 2\delta \\ &< \varepsilon, \end{aligned}$$

so $\gamma_y(t_0 - r) \in D^\varepsilon(\xi)$. Since $y \notin D^\varepsilon(\xi)$, it follows that the geodesic segment $[v_0, y]$ goes through $D^\varepsilon(\xi)$ and leaves it for some $t_1 > t_0 - r$.

Moreover, after leaving $D^\varepsilon(\xi)$ the geodesic $[v_0, y]$ meets the τ-ball centred at $\gamma_x(t_0)$ for some $t_2 \geq t_1$. Indeed, this is clear if $x \in \overline{D^\varepsilon(\xi)}$ since $d(x,y) < \delta \leq \tau$. If $x \notin \overline{D^\varepsilon(\xi)}$, then $[v_0, y]$ meets the 2δ-ball centred at $\gamma_x(t_0+\delta_1)$, which is contained in $(X \setminus D^\varepsilon(\xi)) \cap B(\gamma_x(t_0), 2\delta_1)$ by definition of δ . Hence, $[v_0, y]$ meets $B(\gamma_x(t_0), \tau) \setminus D^\varepsilon(\xi)$ for some $t_2 \geq t_1$.

We thus have $d(\gamma_x(t_0 - r), \gamma_y(t_0 - r)) \leq \tau$ and $d(\gamma_x(t_0), \gamma_y(t_2)) \leq \tau$. By definition of τ, it follows that

$$\gamma_y\Bigg([t_0 - r, t_2] \Bigg) \subset \mathrm{st}\left(\sigma_{\xi,\varepsilon}(x)\right),$$

which implies $\sigma_{\xi,\varepsilon}(y) \subset \mathrm{st}\big(\sigma_{\xi,\varepsilon}(x)\big)$. □

The star lemma IV.4.11 immediately implies the following:

Corollary IV.4.12. *Let $\xi \in \partial_{Stab}G$, $\mathcal{U}$ a ξ-family and $\varepsilon \in (0,1)$. Then the sets $Cone_{\mathcal{U},\varepsilon}(\xi)$ and $\widetilde{Cone}_{\mathcal{U},\varepsilon}(\xi)$ are open in X.* □

IV.5 The topology of $\overline{EG}$.

In this section, we define a topology on $\overline{EG}$ and study its first properties.

IV.5.1 Definition of the topology.

In this paragraph, we define a topology on $\overline{EG}$, by defining a basis of open neighbourhoods at every point. Since points of $\overline{EG}$ are of three different kinds (EG, ∂X and $\partial_{Stab}G$), we treat these cases separately.

Definition IV.5.1. Let $\widetilde{x} \in EG$. We define a basis of neighbourhoods of $\widetilde{x}$ in $\overline{EG}$, denoted $\mathcal{O}_{\overline{EG}}(\widetilde{x})$, as the set of open sets of EG containing $\widetilde{x}$.

We now turn to the case of points of the boundary of X. Recall that since X is a simplicial CAT(0) space with countably many simplices, the bordification $\overline{X} = X \cup \partial X$ has a natural metrisable topology, though not necessarily compact if X is not locally finite. For every $\eta \in \partial X$, a basis of neighbourhoods of η in that bordification is given by the family of

$$V_{r,\delta}(\eta) = \left\{ x \in \overline{X} \middle| d(v_0, x) > r \text{ and } \gamma_x(r) \in B(\gamma_\eta(r), \delta) \right\}, \ r, \delta > 0.$$

Remark: For $r, \delta > 0$, $\eta \in \partial X$ and if γ is the parametrisation of a geodesic such that there exists $T \geq 0$ with $\gamma(T) \in V_{r,\delta}(\eta)$, then $\gamma(t) \in V_{r,\delta}(\eta)$ for every $t \geq T$.

We denote by $\mathcal{O}_{\overline{X}}(\eta)$ this basis of neighbourhoods of η in $\overline{X}$. Endowed with that topology, $\overline{X}$ is a second countable metrisable space (see [9]).

Note that the topology of $\overline{X}$ satisfies the following properties:

Lemma IV.5.2. *Let $\eta \in \partial X$. Then there exists a basis of neighbourhoods (U_n) of η in $\overline{X}$ such that U_n and $U_n \setminus \partial X$ are contractible for every $n \geq 0$.*

Proof. For $r, \delta > 0$, let $U_{r,\delta}(\eta) = V_{r,\delta}(\eta) \cup B(\gamma_\eta(r), \delta)$. This defines a basis of neighbourhoods of η in $\overline{X}$. As $U_{r,\delta}(\eta) \setminus \partial X$ can be retracted by strong deformation along geodesics starting at v_0 onto $B(\gamma_\eta(r), \delta)$, it is contractible. Furthermore, as $U_{r,\delta}(\eta)$ can be retracted by strong deformation onto $U_{r,\delta}(\eta) \setminus \partial X$, the same holds for $U_{r,\delta}(\eta)$. □

Lemma IV.5.3. *Let $\eta \in \partial X$, U a neighbourhood of η in $\overline{X}$ and $k \geq 0$. Then there exists a neighbourhood U' of η in $\overline{X}$ that is contained in U and such that $d(U' \cap X, X \setminus U) > k$.*

Proof. The definition of the topology of $\overline{X}$ implies the following: if (x_n) and (y_n) are two sequences of X such that $d(x_n, y_n)$ is bounded, then (x_n) converges to a point of ∂X if and only if (y_n) converges to the same point. Reasoning by contradiction thus implies the lemma. □

Definition IV.5.4. Let $\eta \in \partial X$, and let U be a neighbourhood of η in $\overline{X}$. We set

$$V_U(\eta) = p^{-1}(U \cap X) \cup (U \cap \partial X) \cup \{\xi \in \partial_{Stab} G | D(\xi) \subset U\}.$$

When U runs over the basis $\mathcal{O}_{\overline{X}}(\eta)$ of neighbourhoods of η in $\overline{X}$, the above formula defines a collection of neighbourhoods for η in $\overline{EG}$, denoted $\mathcal{O}_{\overline{EG}}(\eta)$.

We finally define open neighbourhoods for points in $\partial_{Stab} G$.

Definition IV.5.5. Let $\xi \in \partial_{Stab} G, \mathcal{U} \subset \mathcal{U}_\xi$ be a ξ-family, and $\varepsilon \in (0, 1)$. A neighbourhood $V_{\mathcal{U},\varepsilon}(\xi)$ is defined in four steps as follows:

- Let $W_{\mathcal{U},\varepsilon}(\xi)$ be the set of points $\widetilde{x} \in EG$ whose projection $x \in X$ belongs to $D^\varepsilon(\xi)$ and is such that for some (hence every) vertex v of $D(\xi) \cap \sigma_x$, we have $\phi_{v,\sigma_x}(\widetilde{x}) \in U_v$.
- Let W_1 be the set of points of EG whose projection in X belongs to $\mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$.
- Let W_2 be the set of points of ∂X that belong to $\mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$.
- Let W_3 be the set of points $\xi' \in \partial_{Stab} G$ such that $D(\xi') \setminus D(\xi) \subset \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$ and for every vertex v of $D(\xi) \cap D(\xi')$ we have $\xi' \in U_v$.

Now set

$$V_{\mathcal{U},\varepsilon}(\xi) = W_{\mathcal{U},\varepsilon}(\xi) \cup W_1 \cup W_2 \cup W_3.$$

This collection of neighbourhoods of ξ in $\overline{EG}$ is denoted $\mathcal{O}_{\overline{EG}}(\xi)$. Note that these neighbourhoods depend on the chosen basepoint v_0. If we need to specify the basepoint used to define the various sets $\mathrm{Cone}_{\mathcal{U},\varepsilon}(\xi)$, $V_{\mathcal{U},\varepsilon}(\xi)$, we will indicate it in superscript. In that case, we will speak of the topology (of $\overline{EG}$) *centred* at a given point.

Note that for ξ-families $\mathcal{U}' \subset \mathcal{U}$ and $\varepsilon' < \varepsilon$, we do not necessarily have the inclusion $V_{\mathcal{U}',\varepsilon'}(\xi) \subset V_{\mathcal{U},\varepsilon}(\xi)$ since these two neighbourhoods are defined by looking at the way geodesics leave two (a priori non related) different neighbourhoods of the domain $D(\xi)$. However, the crossing lemma IV.4.4 immediately implies the following:

Lemma IV.5.6. *Let $\xi \in \partial_{Stab}G$, $\mathcal{U}$,$\mathcal{U}'$ two ξ-families, and $0 < \varepsilon' < \varepsilon$. If $\mathcal{U}'$ is d_ξ-nested in $\mathcal{U}$, then $V_{\mathcal{U}',\varepsilon'}(\xi) \subset V_{\mathcal{U},\varepsilon}(\xi)$.* □

Definition IV.5.7. We define a topology on $\overline{EG}$ by taking the topology generated by the elements of $\mathcal{O}_{\overline{EG}}(x)$, for every $x \in \overline{EG}$. We denote by $\mathcal{O}_{\overline{EG}}$ the set of elements of $\mathcal{O}_{\overline{EG}}(x)$ when x runs over $\overline{EG}$. Thus, any an open set in $\overline{EG}$ is a union of finite intersections of elements of $\mathcal{O}_{\overline{EG}}$.

We will show in the next subsection that $\mathcal{O}_{\overline{EG}}$ is actually a *basis* for the topology of $\overline{EG}$.

IV.5.2 A basis of neighbourhoods.

Here we prove that the set of neighbourhoods we just defined is a basis for the topology of $\overline{EG}$. In order to do that, we need the following:

Filtration Lemma. Let $z, z' \in \overline{EG}$ and $U \in \mathcal{O}_{\overline{EG}}(z)$ an open neighbourhood of z. If $z' \in U$, then there exists an open neighbourhood of z', $U' \in \mathcal{O}_{\overline{EG}}(z')$, such that $U' \subset U$.

Since points of $\overline{EG}$ are of three different natures (EG, ∂X, and $\partial_{Stab}G$), the proof breaks into six distinct cases. We first introduce a notation that will be useful to treat similar cases at once.

Definition IV.5.8. We extend the projection $p : EG \to X$ to a map $\bar{p}$ from $\overline{EG}$ to the set of subsets of $\overline{X}$ in the following way:

- For $\widetilde{x} \in EG$, we define $\bar{p}(z)$ to be the singleton $\{p(\widetilde{x})\}$.
- For $\eta \in \partial X$, we define $\bar{p}(\eta)$ to be the singleton $\{\eta\}$.
- For $\xi \in \partial_{Stab}G$, we set $\bar{p}(\xi) = D(\xi)$.
- Finally, for $K \subset \overline{EG}$, we set $\bar{p}(K) = \bigcup_{z \in K} \bar{p}(z)$.

Lemma IV.5.9. *Let $\widetilde{x}, \widetilde{y} \in EG$ and $U \in \mathcal{O}_{EG}(\widetilde{x})$ an open neighbourhood of $\widetilde{x}$ in EG. If $\widetilde{y} \in U$, then there exists an open neighbourhood of $\widetilde{y}$ in EG, $U' \in \mathcal{O}_{\overline{EG}}(\widetilde{y})$ such that $U' \subset U$.*

Proof. By definition of the topology, we can take $U' = U$. □

Lemma IV.5.10. *Let $\eta, \eta' \in \partial X$ and $U \in \mathcal{O}_{\overline{X}}(\eta)$ an open neighbourhood of η in $\overline{X}$. If $\eta' \in V_U(\eta)$, then there exists an open neighbourhood U' of η' in $\overline{X}$, such that $V_{U'}(\eta') \subset V_U(\eta)$.*

Proof. Since $\mathcal{O}_{\overline{X}}$ is a basis of neighbourhoods for $\overline{X}$, there exists a neighbourhood $U' \in \mathcal{O}_{\overline{X}}(\eta')$ such that $U' \subset U$. Now one clearly has $\eta \in V_{U'}(\eta') \subset V_U(\eta)$. □

Lemma IV.5.11. *Let $\widetilde{x} \in EG, \eta \in \partial X$ and U an open neighbourhood of η in $\overline{X}$. If $\widetilde{x} \in V_U(\eta)$, then there exists an open neighbourhood U' of $\widetilde{x}$ in $\overline{EG}$, $U' \in \mathcal{O}_{EG}(\widetilde{x})$, such that $U' \subset V_U(\eta)$.*

Proof. It is enough to choose an arbitrary open neighbourhood U' of $\widetilde{x}$ contained in $p^{-1}(U \cap X)$. □

Lemma IV.5.12. *Let $\xi \in \partial_{Stab}G, \eta \in \partial X$ and $U \in \mathcal{O}_{\overline{X}}(\eta)$ an open neighbourhood of η in $\overline{X}$. If $\xi \in V_U(\eta)$, then there exist $\varepsilon \in (0,1)$ and a ξ-family $\mathcal{U}$ such that $V_{\mathcal{U},\varepsilon}(\xi) \subset V_U(\eta)$.*

Proof. The subcomplex $D(\xi) \subset U$ is finite, hence compact, so choose $\varepsilon \in (0,1)$ such that $D^\varepsilon(\xi) \subset U$. Let $\mathcal{U}$ be any ξ-family. For every $x \in \widetilde{\text{Cone}}_{\mathcal{U},\varepsilon}(\xi)$, the geodesic segment $[v_0, x]$ meets $D(\xi)$ by Corollary IV.4.9. As $D(\xi)$ is contained in U, the same holds for x. It then follows that $V_{\mathcal{U}_\xi,\varepsilon}(\xi) \subset V_U(\eta)$. □

Lemma IV.5.13. *Let $\eta \in \partial X, \xi \in \partial_{Stab}G$, $\mathcal{U}$ a ξ-family and $\varepsilon \in (0,1)$. If $\eta \in V_{\mathcal{U},\varepsilon}(\xi)$, then there exists an open neighbourhood U of η in $\overline{X}$ such that $V_U(\eta) \subset V_{\mathcal{U},\varepsilon}(\xi)$.*

Proof. Let $\gamma_\eta : [0,\infty) \to X$ be a parametrisation of the geodesic ray $[v_0, \eta)$. The subcomplex $D(\xi)$ being finite by Proposition IV.3.2, choose $R > 0$ such that $D(\xi) \subset B(v_0, R)$, and let $x = \gamma_\eta(R+1)$. Since $\eta \in V_{\mathcal{U},\varepsilon}(\xi)$, we have $x \in \text{Cone}_{\mathcal{U},\varepsilon}(\xi)$, which is open in X by Corollary IV.4.12. Let $\delta > 0$ such that $B(x,\delta) \subset \text{Cone}_{\mathcal{U},\varepsilon}(\xi)$. Now if we set $U' = V_{R+1,\delta}(\eta) \in \mathcal{O}_{\overline{X}}(\eta)$, it follows that $V_{U'}(\eta) \subset V_{\mathcal{U},\varepsilon}(\xi)$. □

Lemma IV.5.14. *Let $\widetilde{x} \in EG, \xi \in \partial_{Stab}G$, $\mathcal{U}$ a ξ-family and $\varepsilon \in (0,1)$. If $\widetilde{x} \in V_{\mathcal{U},\varepsilon}(\xi)$, then there exists a $U \in \mathcal{O}_{\overline{EG}}(\widetilde{x})$ such that $U \subset V_{\mathcal{U},\varepsilon}(\xi)$.*

Proof. It is enough to prove that $V_{\mathcal{U},\varepsilon}(\xi) \cap EG$ is open in EG. First, since the maps $\phi_{\sigma,\sigma'}$ are embeddings, it is clear that $W_{\mathcal{U},\varepsilon}(\xi)$ is open in EG. Let $\widetilde{y} \in V_{\mathcal{U},\varepsilon}(\xi) \cap EG$ with $y = p(\widetilde{y}) \notin D^\varepsilon(\xi)$. The star lemma IV.4.11 yields a $\delta > 0$ such that for every $z \in B(y,\delta) \setminus D^\varepsilon(\xi)$, the geodesic segment $[v_0, z]$ goes through $D^\varepsilon(\xi)$ and $\sigma_{\xi,\varepsilon}(z) \subset \text{st}(\sigma_{\xi,\varepsilon}(y))$. We can further assume that $B(y,\delta) \subset \text{st}(\sigma_y)$. It now follows immediately from the construction of $V_{\mathcal{U},\varepsilon}(\xi)$ that $p^{-1}(B(y,\delta))$ is an open neighbourhood of $\widetilde{x}$ contained in $V_{\mathcal{U},\varepsilon}(\xi)$, which concludes the proof. □

Lemma IV.5.15. *Let $\xi, \xi' \in \partial_{Stab}G$, $\mathcal{U}$ a ξ-family and $\varepsilon \in (0,1)$. If $\xi' \in V_{\mathcal{U},\varepsilon}(\xi)$, then there exists a ξ'-family $\mathcal{U}'$ and $\varepsilon' \in (0,1)$ such that $V_{\mathcal{U}',\varepsilon'}(\xi') \subset V_{\mathcal{U},\varepsilon}(\xi)$.*

By Lemma IV.4.11, let $\delta \in (0,\varepsilon)$ be such that for all $y \in D^\delta(\xi') \setminus D^\varepsilon(\xi)$, the geodesic segment $[v_0, y]$ goes through $D^\varepsilon(\xi)$ and is such that $\sigma_{\xi,\varepsilon}(y) \subset \text{st}(\sigma_{\xi,\varepsilon}(x))$, for some $x \in D(\xi')$. We now define a ξ'-family using the following lemma.

Lemma IV.5.16. *There exist nested ξ'-families $\mathcal{U}^{[d_\xi]} \supset \ldots \supset \mathcal{U}^{[0]} = \mathcal{U}'$ such that the following holds: Let x be a point of $Cone_{\mathcal{U}',\delta}(\xi')$ such that the geodesic from v_0 to x leaves $D^\delta(\xi')$ at a point which is still inside $D^\varepsilon(\xi)$. Let $\sigma_1 = \sigma_{\xi',\delta}(x), \ldots, \sigma_n = \sigma_{\xi,\varepsilon}(x)$ ($n \leq d_\xi$) be the path of simplices met by the geodesic segment $[v_0, x]$ inside $\overline{D^\varepsilon}(\xi)$ after leaving $D^\delta(\xi')$ (cf Figure IV.4).*

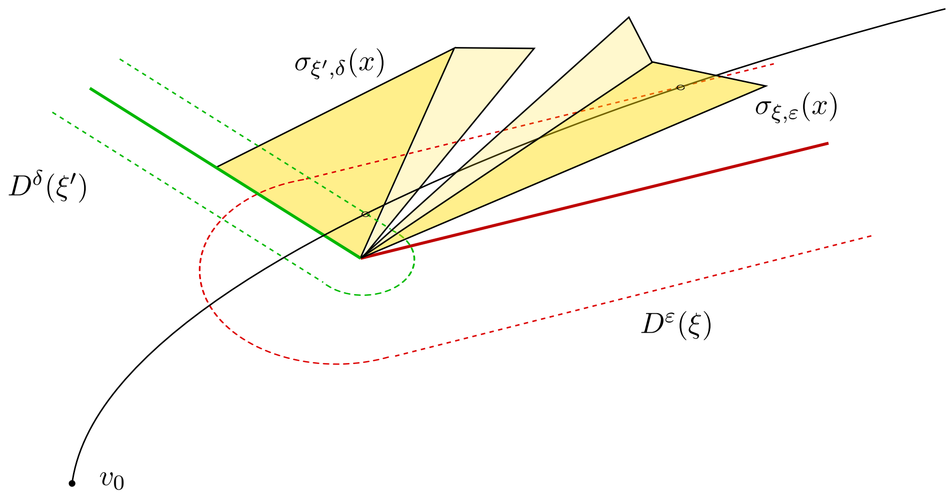

Figure IV.4.

We then have the following, for every $1 \leq k \leq n$:

(i) The simplex σ_k is contained in $\bigcup_{v' \in V(\xi) \cap V(\xi')} st(v')$ but not in $\bigcup_{v \in V(\xi) \setminus V(\xi')} st(v)$.

(ii) For every vertex v' of σ_k contained in $D(\xi')$, the inclusion $\overline{EG_{\sigma_k}} \subset U_{v'}^{[k]}$ holds in $\overline{EG_{v'}}$.

Proof. If v' is a vertex of $D(\xi) \cap D(\xi')$, then for every vertex v of $\overline{st}(v') \cap (D(\xi) \setminus D(\xi'))$, choose a neighbourhood $W_{v,v'}$ of ξ' in $\overline{EG_{v'}}$ missing $\overline{EG}_{[v,v']}$, and set

$$W_{v'} = \left(\bigcap_{v \in \overline{st}(v') \cap (V(\xi) \setminus V(\xi'))} W_{v,v'} \right) \cap U_{v'}.$$

If v' is a vertex not in $D(\xi)$, set $W_v = \overline{EG_{v'}}$.
We now define $\mathcal{U}'$ to be a ξ'-family that is d_ξ-nested in the family of $W_{v'}, v' \in D(\xi')$, that is, $\mathcal{U}'$ is a ξ'-family such that there exists a sequence of nested ξ'-families $\mathcal{U}^{[d_\xi]} \supset \ldots \supset \mathcal{U}^{[0]} = \mathcal{U}'$ satisfying $W_{v'} \supset U_{v'}^{[d_\xi]} \supset \ldots \supset U_{v'}^{[0]} = U'_{v'}$ for every vertex v' of $D(\xi')$.

We now prove (i) and (ii) by induction on k. Since the geodesic segment $[v_0, x]$ leaves $D^\delta(\xi')$ while inside $D^\varepsilon(\xi)$, we have $\sigma_1 = \sigma_{\xi',\delta}(x) \subset \bigcup_{v' \in V(\xi) \cap V(\xi')} \mathrm{st}(v')$. To prove (i) for

$k=1$, we reason by contradiction. Suppose there exists a vertex v' of $D(\xi)\cap D(\xi')$ and a vertex v of $D(\xi)\setminus D(\xi')$ such that $\sigma_1\subset \mathrm{st}\big([v,v']\big)$, then we have $\overline{EG}_{\sigma_1}\subset \overline{EG}_{[v,v']}$ in $\overline{EG}_{v'}$. But the former set is contained in $U_{v'}$ since $\widetilde{x}\in V_{\mathcal{U}',\delta}(\xi')$, and the latter is disjoint from $U_{v'}$ by construction of $\mathcal{U}'$, which is absurd.

Suppose the result has been proved up to rank k. If $\sigma_{k+1}\subset\sigma_k$, the result is straightforward, so we suppose that $\sigma_k\subset\sigma_{k+1}$. We prove (i) by contradiction. Suppose there exists a vertex v' of $D(\xi)\cap D(\xi')$ and a vertex v of $D(\xi)\setminus D(\xi')$ such that $\sigma_{k+1}\subset \mathrm{st}\big([v,v']\big)$. Then by the induction hypothesis, we have $\overline{EG}_{[v,v']}\cap U_{v'}^{[k]}\neq\varnothing$ in $\overline{EG}_{v'}$, hence $\overline{EG}_{[v,v']}\subset U_{v'}^{[k+1]}\subset W_{v'}$ since $\mathcal{U}^{[k]}$ is nested in $\mathcal{U}^{[k+1]}$,and the last inclusion contradicts the definition of $\mathcal{U}'$.

We now prove (ii). Let v_k a vertex of $D(\xi)\cap D(\xi')$ contained in σ_k (hence in σ_{k+1}). Thus we have $\overline{EG}_{\sigma_{k+1}}\subset\overline{EG}_{\sigma_k}\subset U_{v_k}^{[k]}\subset U_{v_k}^{[k+1]}$ in $\overline{EG}_{v_k}$. Now let v' be another vertex of $D(\xi')\cap D(\xi)$ contained in σ_{k+1} (if any). We thus have $\overline{EG}_{[v_k,v']}\cap U_{v_k}^{[k]}\neq\varnothing$ in $\overline{EG}_{v_k}$, so $\overline{EG}_{[v_k,v']}\subset U_{v_k}^{[k+1]}$ in $\overline{EG}_{v_k}$. But by Proposition IV.3.12, this implies $\overline{EG}_{[v_k,v']}\subset U_{v'}^{[k+1]}$, which proves (ii). □

Proof of Lemma IV.5.15. Let us show now that $V_{\mathcal{U}',\delta}(\xi')\subset V_{\mathcal{U},\varepsilon}(\xi)$. Let $z\in V_{\mathcal{U}',\delta}(\xi')$ and $x\in\bar{p}(z)$. The geodesic $[v_0,x]$ meets $D^\delta(\xi')$, hence $D^\varepsilon(\xi)$. To prove that $z\in V_{\mathcal{U},\varepsilon}(\xi)$, it is now enough to prove that $x\in\widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$.

If $x\in W_{\mathcal{U}',\delta}(\xi')\cap D^\varepsilon(\xi)$, it follows from the definition of $\mathcal{U}'$ (defined in IV.5.16) that $z\in W_{\mathcal{U},\varepsilon}(\xi)$.

If the geodesic segment $[v_0,x]$ meets $D^\delta(\xi')$ outside $D^\varepsilon(\xi)$, it follows from the definition of δ that there exists $x'\in D(\xi')\setminus D(\xi)$ such that $\sigma_{\xi,\varepsilon}(x)\subset st\left(\sigma_{\xi,\varepsilon}(x')\right)$. But since $x'\in\widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$, the same holds for x.

Thus the only case left to consider is when the geodesic segment $[v_0,x]$ leaves $D^\delta(\xi')$ while still being inside $D^\varepsilon(\xi)$. But by the previous lemma, we get that for every vertex v' of $\sigma_{\xi,\varepsilon}(x)$ contained in $D(\xi)$, $\overline{EG}_{\sigma_{\xi,\varepsilon}(x)}\subset U_v^{[n]}\subset U_v$ in $\overline{EG}_v$, which now implies $x\in\widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$. This concludes the proof. □

Theorem IV.5.17. *$\mathcal{O}_{\overline{EG}}$ is a basis for the topology of $\overline{EG}$, which makes it a second countable space. For this topology, EG embeds as a dense open subset.*

Proof. To prove that $\mathcal{O}_{\overline{EG}}$ is a basis for the topology of $\overline{EG}$, it is enough to show that for every open sets U_1,U_2 of $\overline{EG}$ and every $z\in U_1\cap U_2$, there exists an open neighbourhood $W\in\mathcal{O}_{\overline{EG}}$ such that $z\in W\subset U_1\cap U_2$.

If $z\in EG$: By the results from the previous paragraph, there exists $V_1,V_2\in\mathcal{O}_{EG}(z)$ such that $V_1\subset U_1$ and $V_2\subset U_2$. Then take W to be any element of $\mathcal{O}_{EG}(z)=\mathcal{O}_{\overline{EG}}(z)$ contained in $V_1\cap V_2$.

If $z=\eta\in\partial X$: By the results from the previous paragraph, let $O_1,O_2\in\mathcal{O}_{\overline{X}}(\eta)$ such that $V_{O_1}(\eta)\subset U_1$ and $V_{O_2}(\eta)\subset U_2$. Choosing a neighbourhood $W\in\mathcal{V}_{\overline{X}}(\eta)$ contained in $O_1\cap O_2$, it follows that $V_W(\eta)\subset U_1\cap U_2$.

If $z = \xi \in \partial_{Stab}G$: By the results from the previous paragraph, let $V_{\mathcal{U}_1,\varepsilon_1}(\xi), V_{\mathcal{U}_2,\varepsilon_2}(\xi)$ such that $V_{\mathcal{U}_1,\varepsilon_1}(\xi) \subset U_1$ and $V_{\mathcal{U}_2,\varepsilon_2}(\xi) \subset U_2$. Let $\mathcal{U}$ be a ξ-family which is d_ξ-nested in $\{(U_1)_v \cap (U_2)_v, v \in V(\xi)\}$, and let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. It follows from Lemma IV.5.6 that $V_{\mathcal{U},\varepsilon}(\xi) \subset V_{\mathcal{U}_1,\varepsilon_1}(\xi) \cap V_{\mathcal{U}_2,\varepsilon_2}(\xi) \subset U_1 \cap U_2$.

To prove that this topology is second countable, we define countable many open sets $(U_n)_{n\geq 0}$ such that for every open set U in $\mathcal{O}_{\overline{EG}}$ and every x in U, there exist an integer m such that $x \in U_m \subset U$.

Since EG is the realisation of a complex of spaces over a simplicial complex with countably many simplices, and with fibres that have a CW-structure with countably many cells, it inherits a CW-complex structure with countably many cells. Thus its topology is second countable, and we can choose a countable basis of neighbourhoods $(U_n), n \geq 0$ of EG.

Since X is a simplicial complex with countably many cells, it is a separable space, hence so is the set Λ of points lying on a geodesic from v_0 to a point of ∂X (note that a given geodesic segment may not necessarily be extendable to a geodesic ray). Let Λ' be a dense countable subset of Λ. Now the family of open sets $V_{r,\varepsilon}(\eta)$ for $\eta \in \partial X$, $\gamma_\eta(r) \in \Lambda'$ and $\varepsilon \in \mathbb{Q}$ is a countable family, yielding a countable family of open neighbourhoods of $\overline{EG}$, denoted $(V_n)_{n\geq 0}$. Note that $(V_n)_{n\geq 0}$ contains a basis of neighbourhoods for every point of $\overline{EG}$ that belongs to ∂X.

A neighbourhood of a point ξ of $\partial_{Stab}G$ is defined by choosing a constant $\varepsilon \in (0,1)$, a finite subcomplex of X (the domain of ξ), and for every vertex v of that subcomplex an open set of $\overline{EG_v}$. Since domains of points of $\partial_{Stab}G$ are finite by Proposition IV.3.2, there are only countably many such subcomplexes. Furthermore, for every vertex v of X, $\overline{EG_v}$ has a countable basis of neighbourhoods. It is now clear that we can define a countable family $(W_n)_{n\geq 0}$ of open neighbourhoods, containing a basis of neighbourhoods of every element of $\partial_{Stab}G$.

The family consisting of all the U_n, V_n, W_n is now a countable basis of neighbourhoods of $\overline{EG}$.

Finally, the subset EG, which is open by construction of the topology, is dense in $\overline{EG}$ since every open set in that basis of neighbourhoods meets EG by construction. □

Lemma IV.5.18. *The topology of $\overline{EG}$ does not depend on the choice of a basepoint. Moreover, the action of G on EG continuously extends to ∂G.*

Proof. Choose x_0 and x_1 two points of X (note that we do not assume these points to be vertices). Throughout this proof, we will indicate the dependence on the basepoint by indicating it in superscript, as explained in Definition IV.5.5. It is a well known fact that the topology of $\overline{X}$ does not depend on the basepoint, so it is enough to consider neighbourhoods of points in $\partial_{Stab}G$.

Recall that the number of simplices in a domain $D(\xi)$, $\xi \in \partial_{Stab}G$ is uniformly bounded by the constant $d_{\max}$ defined in IV.3.3. Let $\xi \in \partial_{Stab}G$, $\mathcal{U}_0$ a ξ-family for the topology centred at x_0 and $\varepsilon > 0$. Now let $\mathcal{U}_1$ be a ξ-family for the topology centred at x_1, which is $2d_{\max}$-refined in $\mathcal{U}_0$. Let x be a point of $\widetilde{\mathrm{Cone}}^{x_1}_{\mathcal{U}_1,\varepsilon}(\xi)$. Then the geodesic reattachment lemma IV.4.8 implies that $[x_0, x]$ meets $D(\xi)$. We can thus apply Lemma IV.1.7 to subsegments of $[x_0, x]$ and $[x_1, x]$, and to simplices $\sigma^{x_0}_{\xi,\varepsilon}(x)$ and $\sigma^{x_1}_{\xi,\varepsilon}(x)$. Since $\mathcal{U}_1$ is $2d_{\max}$-refined in $\mathcal{U}_0$, it follows that $x \in \widetilde{\mathrm{Cone}}^{x_0}_{\mathcal{U}_0,\varepsilon}(\xi)$, hence $\widetilde{\mathrm{Cone}}^{x_1}_{\mathcal{U}_1,\varepsilon}(\xi) \subset \widetilde{\mathrm{Cone}}^{x_0}_{\mathcal{U}_0,\varepsilon}(\xi)$. Moreover, since $\mathcal{U}_1$ is contained in $\mathcal{U}_0$, we get $V^{x_1}_{\mathcal{U}_1,\varepsilon}(\xi) \subset V^{x_0}_{\mathcal{U}_0,\varepsilon}(\xi)$.

We extend the G-action on EG to ∂G as follows. First note that the action naturally extends to ∂X. Indeed, G acts on the CAT(0) space X by isometries, and those isometries naturally extend to homeomorphisms of the visual boundary ∂X. Furthermore, we defined in Section 2 a G-action on $\partial_{Stab} G$. Thus we have an action of G on $\overline{EG}$, which we now prove to be continuous.

Let $g \in G$. Since EG is open in $\overline{EG}$ and the action of G on EG is continuous, it is enough to check the continuity at points of ∂G. For a point $z \in \partial G$, the element g sends a basis of neighbourhood of z for the topology centred at v_0 to a basis of neighbourhoods of $g.z$ for the topology centred at $g.v_0$. Since the topology does not depend on the basepoint by the above discussion, the action of g is continuous at points of ∂G. □

IV.5.3 Induced topologies.

Proposition IV.5.19. *The topology of $\overline{EG}$ induces the natural topologies on EG, ∂X and $\overline{EG_v}$ for every vertex v of X.*

We first prove that for any open set U in the basis of neighbourhoods $\mathcal{O}_{\overline{EG}}$ previously defined, $U \cap EG$ is open in EG. For $x \in EG$, the result is obvious for points in $\mathcal{O}_{\overline{EG}}(x)$ since open sets in $\mathcal{O}_{\overline{EG}}(x)$ are open sets of EG by definition. For $\eta \in \partial X$ and U a neighbourhood of η in $\overline{X}$, we have $V_U(\eta) \cap EG = p^{-1}(U \cap X)$ which is open in EG. For $\xi \in \partial_{Stab} G$, $\varepsilon \in (0,1)$ and $\mathcal{U}$ a ξ-family, it was proven in Lemma IV.5.14 that $V_{\mathcal{U},\varepsilon}(\xi) \cap EG$ is open in EG.

We now prove that for any open set U in the basis of neighbourhoods $\mathcal{O}_{\overline{EG}}$, $U \cap \partial X$ is open in ∂X. For a point $\eta \in \partial X$ and U a neighbourhood of η in $\overline{X}$, we have $V_U(\eta) \cap \partial X = U \cap \partial X$, which is open in ∂X. Now consider $\xi \in \partial_{Stab} G$, $\varepsilon \in (0,1)$ and $\mathcal{U}$ a ξ-family. If $V_{\mathcal{U},\varepsilon}(\xi) \cap \partial X$ is empty there is nothing to prove, otherwise let $\eta \in V_{\mathcal{U},\varepsilon}(\xi) \cap \partial X$. By Lemma IV.5.13, let U' be a neighbourhood of η in $\overline{X}$ such that $\mathcal{V}_{U'}(\eta) \subset V_{\mathcal{U},\varepsilon}(\xi)$. Thus, $\eta \in U' \cap \partial X \subset V_{\mathcal{U},\varepsilon}(\xi) \cap \partial X$, and $V_{\mathcal{U},\varepsilon}(\xi) \cap \partial X$ is open in ∂X.

Before proving the analogous result for $\overline{EG_v}$, with v a vertex of X, we need the following lemma.

Lemma IV.5.20. *Let $\xi \in \partial_{Stab} G$, $\mathcal{U}$ a ξ-family and $\varepsilon \in (0,1)$. Recall that $d_{\max}$ was defined in IV.3.3 as an integer such that domains of points of $\partial_{Stab} G$ meet at most $d_{\max}$ simplices. Let $\mathcal{U}'$ be a ξ-family which is $d_{\max}$-refined in $\mathcal{U}$. Then we have $\bigcup_{v \in D(\xi)} U'_v \cap \partial G_v \subset V_{\mathcal{U},\varepsilon}(\xi)$.*

Proof. Let $\xi' \in \bigcup_{v\in D(\xi)} U_v' \cap \partial G_v$ and $x \in D(\xi')$. If x is a vertex of $D(\xi) \cap D(\xi')$, the definition of a ξ-family implies that $\xi' \in U_x$. Otherwise, since $D(\xi')$ is convex by Proposition IV.3.2, let γ be a geodesic path in $D(\xi')$ from x to $D(\xi)$ and meeting $D(\xi)$ at a single point. This yields a path of open simplices from a simplex $\sigma \subset \mathrm{N}(D(\xi)) \setminus D(\xi)$ to σ_x of length at most $d_{\max}$ in $D(\xi') \setminus D(\xi)$. Since $\xi' \in \bigcup_{v\in D(\xi)} U_v' \cap \partial G_v$ also belongs to ∂G_σ, we have $\sigma \subset \mathrm{N}_{\mathcal{U}'}(D(\xi))$. Now since $\mathcal{U}'$ is $d_{\max}$-refined in $\mathcal{U}$, we get $\sigma_x \subset \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$ by Lemma IV.4.10. □

Proof of Proposition IV.5.19. Let v be a vertex of X. We now prove that for every open set U in the basis of neighbourhood $\mathcal{O}_{\overline{EG}}$, $U \cap \overline{EG_v}$ is open in $\overline{EG_v}$.

We proved already that the topology of $\overline{EG}$ induces the natural topology on EG. Now using the filtration lemmas IV.5.12 and IV.5.15, it is enough to show, for every $\xi \in \partial G_v$, every $\varepsilon \in (0,1)$ and every ξ-family $\mathcal{U}$, that $V_{\mathcal{U},\varepsilon}(\xi) \cap \overline{EG_v}$ contains a neighbourhood of ξ in $\overline{EG_v}$. By Lemma IV.5.20, let $\mathcal{U}'$ be a ξ-family contained in $\mathcal{U}$ and such that every point of $U_v' \cap \partial G_v$ belongs to $V_{\mathcal{U},\varepsilon}(\xi)$. Then we have $\xi \in U_v' \subset V_{\mathcal{U},\varepsilon}(\xi) \cap \overline{EG_v}$, and so $V_{\mathcal{U},\varepsilon}(\xi) \cap \overline{EG_v}$ is open in $\overline{EG_v}$. Thus the topology of $\overline{EG}$ induces the natural topology on $\overline{EG_v}$.

Finally, note that the map $\overline{EG_v} \to \overline{EG}$ is injective by Proposition IV.3.4. As $\overline{EG_v}$ is a compact space, that map is an embedding. □

In the exact same way, we can prove the following:

Lemma IV.5.21. *Let σ be a closed cell of X. Then the quotient map $\sigma \times \overline{EG_\sigma} \to \overline{EG}$ is continuous.* □

IV.6 Metrisability of $\overline{EG}$.

In this section, we prove that $\overline{EG}$ is a compact metrisable space. Recall that by the classical metrisation theorem, it is enough to prove that $\overline{EG}$ is a second countable Hausdorff regular space (see below for definitions) which is sequentially compact.

IV.6.1 Weak separation

In this paragraph, we prove the following:

Proposition IV.6.1. *The space $\overline{EG}$ satisfies the T_0 condition, that is, for every pair of distinct points, there is an open set of $\overline{EG}$ containing one but not the other.*

Note that this property does not imply that the space is Hausdorff. However, we will prove in the next subsection that $\overline{EG}$ is also *regular*, and it is a common result of point-set topology that a space that is T_0 and regular is also Hausdorff. As usual, the proof of Proposition IV.6.1 splits in many cases.

Lemma IV.6.2. *Let $\widetilde{x}, \widetilde{y}$ be two distinct points of $EG \subset \overline{EG}$. Then $\widetilde{x}$ and $\widetilde{y}$ admit disjoint neighbourhoods.*

Proof. Open sets in EG are open in $\overline{EG}$ by definition. The result thus follows from the fact that EG is a Hausdorff space. □

Lemma IV.6.3. *Let η, η' be two distinct points of $\partial X \subset \overline{EG}$. Then η and η' admit disjoint neighbourhoods.*

Proof. The space $\overline{X}$ is metrisable, hence Hausdorff. Choosing disjoint neighbourhoods U of η in $\overline{X}$ (resp. U' of η' in $\overline{X}$) yield disjoint neighbourhoods $V_U(\eta), V_{U'}(\eta')$. □

Lemma IV.6.4. *Let $\widetilde{x} \in EG$ and $\eta \in \partial X$. Then $\widetilde{x}$ and η admit disjoint neighbourhoods.*

Proof. Let $x = p(\widetilde{x}) \in X$. Since $\overline{X}$ is a Hausdorff space, let U be a neighbourhood of x in $\overline{X}$ and U' be a neighbourhood of η' in $\overline{X}$ that are disjoint. Then $p^{-1}(U)$ is a neighbourhood of $\widetilde{x}$ in $\overline{EG}$ and $V_{U'}(\eta)$ is a neighbourhood of η in $\overline{EG}$ that is disjoint from $p^{-1}(U)$. □

Lemma IV.6.5. *Let $\xi \in \partial_{Stab}G$ and $\eta \in \partial X$. Then there exists a neighbourhood of η in $\overline{EG}$ that does not contain ξ.*

Proof. Since $D(\xi)$ is bounded, let $R > 0$ such that the $D(\xi)$ is contained in the R-ball centred at v_0. Now take a neighbourhood U of η in $\overline{X}$ that does not meet that R-ball. The subset $V_U(\eta)$ is a neighbourhood of η in $\overline{EG}$ to which ξ does not belong. □

Lemma IV.6.6. *Let $\widetilde{x} \in EG$ and $\xi \in \partial_{Stab}G$. Then there exists a neighbourhood of $\widetilde{x}$ in $\overline{EG}$ that does not contain ξ.*

Proof. Choose any neighbourhood of $\widetilde{x}$ in EG. This is by definition a neighbourhood of $\widetilde{x}$ in $\overline{EG}$, to which ξ does not belong. □

Lemma IV.6.7. *Let ξ, ξ' be two different points of $\partial_{Stab}G$. Then there exists a neighbourhood of ξ in $\overline{EG}$ that does not contain ξ'.*

Proof. If $D(\xi) \cap D(\xi') \neq \varnothing$, let v be a vertex in that intersection and let U_v be a neighbourhood of ξ in $\overline{EG_v}$ that does not contain ξ'. Now we can take a ξ-family $\mathcal{U}'$ small enough so that $U'_v \subset U_v$ and thus $\xi' \notin V_{\mathcal{U}', \frac{1}{2}}(\xi)$ by Proposition IV.5.19.

If $D(\xi) \cap D(\xi') = \varnothing$, let $x \in D(\xi')$. There are two cases to consider:

- If $[v_0, x]$ does not meet $D(\xi)$, then $V_{\mathcal{U}_\xi, \frac{1}{2}}(\xi)$ does not contain ξ' by Corollary IV.4.9.
- Otherwise, $[v_0, x]$ meets $D(\xi)$ and leaves it. Let σ be the first simplex touched by $[v_0, x]$ after leaving $D(\xi)$, v a vertex of $\sigma \cap D(\xi)$ and U_v a neighbourhood of ξ in $\overline{EG_v}$ that does not contain $\overline{EG_\sigma}$. Now let $\mathcal{U}'$ be ξ-family such that $U'_v \subset U_v$ and $\mathcal{U}''$ a ξ-family that is d_ξ-nested in $\mathcal{U}'$. It then follows from the crossing lemma IV.4.4 that $\xi' \notin V_{\mathcal{U}'', \frac{1}{2}}(\xi)$. □

IV.6.2 Regularity

In this paragraph, we prove the following:

Proposition IV.6.8. *The space $\overline{EG}$ is regular, that is, for every open set U in $\overline{EG}$ and every point $x \in U$, there exists another open set U' containing x and contained in U, and such that every point of $\overline{EG} \setminus U$ admits a neighbourhood that does not meet U'.*

Since we previously defined a basis of neighbourhoods for $\overline{EG}$, it is enough to prove such a proposition for open sets U in that basis. As usual, the proof of Proposition IV.6.8 splits in many cases, depending on the nature of the open sets U and points of U involved.

Lemma IV.6.9. *Let $\widetilde{x} \in EG$ and U an open neighbourhood of $\widetilde{x}$ in $\overline{EG}$. Then there exists a subneighbourhood U' of $\overline{EG}$ containing $\widetilde{x}$ and such that every point in $\overline{EG} \setminus U$ admits a neighbourhood that does not meet U'.*

Proof. The space EG being a CW-complex, its topology is regular, so we can choose a neighbourhood U' of $\widetilde{x}$ in EG whose closure (in EG) is contained in U. Let us call V that closure, and let $x = p(\widetilde{x})$. Since EG is locally finite, we can further assume that $p(V)$ meets only finitely many simplices and that it is contained in $\text{st}(\sigma_x)$. We now show that V is closed in $\overline{EG}$, which implies the proposition.

A point of $EG \setminus V$ clearly admits a neighbourhood in $\overline{EG}$ that does not meet V, since open subsets of EG are open in $\overline{EG}$. For a point $\eta \in \partial X$, choosing any neighbourhood of η in $\overline{X}$ that does not meet $p(V)$ yields a neighbourhood of η in $\overline{EG}$ not meeting V. Thus the only case left is that of a point $\xi \in \partial_{Stab} G$. There are two cases to consider:

If $x \in D(\xi)$, then since $p(V)$ meets only finitely many simplices, it is easy to find a ξ-family $\mathcal{U}$ such that $W_{\mathcal{U},\frac{1}{2}}(\xi)$ misses V, which implies that the whole $V_{\mathcal{U},\frac{1}{2}}(\xi)$ misses V.

If $x \notin D(\xi)$, then Lemma IV.1.5 ensures the existence of a finite subcomplex $K \subset X$ containing $\text{Geod}(v_0, p(V))$. We define a ξ-family $\mathcal{U}$ and a constant ε as follows. Let v be a vertex of $D(\xi)$. For every $\sigma \subset (\text{st}(v) \cap K) \setminus D(\xi)$, let $U_{v,\sigma}$ be a neighbourhood of ξ in $\overline{EG_v}$ which is disjoint from $\overline{EG_\sigma}$. We now set

$$U_v = \bigcap_{\sigma \subset (\text{st}(v) \cap K) \setminus D(\xi)} U_{v,\sigma}.$$

Let $\mathcal{U}$ be a ξ-family which is contained in $\{U_v, v \in V(\xi)\}$, and choose

$$\varepsilon = \min\Big(\frac{1}{3}\text{dist}(p(V), D(\xi)), 1\Big),$$

which is positive since $p(V) \subset \text{st}(\sigma_x)$.

We now show by contradiction that $V_{\mathcal{U},\varepsilon}(\xi) \cap V = \varnothing$. Suppose there exists a point $\widetilde{y}$ in that intersection and let $y = p(\widetilde{y})$. By Corollary IV.4.9, $[v_0, y]$ goes through $D(\xi)$. But since $\widetilde{y} \in V$, we have $\sigma_{\xi,\varepsilon}(y) \subset K$, which contradicts the construction of $\mathcal{U}$.

Thus every point of $\overline{EG} \setminus V$ admits a neighbourhood missing V, so V is closed in $\overline{EG}$. □

Lemma IV.6.10. *Let $\eta \in \partial X$ and U be an open neighbourhood of η in $\overline{X}$. Then there exists an open neighbourhood U' of η in $\overline{X}$ such that every point not in $V_U(\eta)$ admits a neighbourhood that does not meet $V_{U'}(\eta)$.*

Proof. By Lemma IV.5.3, we first choose a neighbourhood W of η in $\overline{X}$ contained in U and such that $d(W \cap X, X \setminus U) > A+1$, where A is the acylindricity constant. Since $\overline{X}$ is metrisable, hence regular, we can further assume that $\overline{W} \subset U$. Finally, we can choose $R > 0$ and $\delta > 0$ such that $U' = V_{R,\delta}(\eta)$ is contained in W and $B(\gamma_\eta(R), \delta)$ is contained in the open star of the minimal simplex containing $\gamma_\eta(R)$ (recall that γ_η is a parametrisation of the geodesic ray $[v_0, \eta)$). We now show that every point not in $V_U(\eta)$ admits a neighbourhood that does not meet $V_{U'}(\eta)$.

Let $z \in EG \setminus V_U(\eta)$. Then $p(z)$ is not in U, hence not in $\overline{U'}$. Since $\overline{U'}$ is closed in $\overline{X}$, there exists an open set U'' of $\overline{X}$ containing $p(z)$ and such that $U'' \subset X \setminus \overline{U'}$. Then $p^{-1}(U'')$ is open in $\overline{EG}$ and $p^{-1}(U'')$ does not meet $V_U(\eta)$.

Let $\eta' \in \partial X \setminus V_U(\eta)$. Then $\eta' \notin U \cap \partial X$ hence $\eta' \notin \overline{U'}$. Since $\overline{U'}$ is closed in $\overline{X}$, we choose an open set U'' in $\mathcal{O}_{\overline{X}}(\eta)$ disjoint from U'. It is now clear that $V_{U''}(\eta')$ does not meet $V'_{\mathcal{U}}(\eta)$.

Let $\xi \in (\partial_{Stab} G) \setminus V_U(\eta)$. To find a neighbourhood of ξ that does not meet $V_{U'}(\eta)$, is enough to find a ξ-family $\mathcal{U}'$ such that $U' \cap \widetilde{\mathrm{Cone}}_{\mathcal{U}', \frac{1}{2}}(\xi) = \varnothing$. We define such a ξ-family as follows:

Let $x = \gamma_\eta(R)$. By Lemma IV.1.5, define a finite subcomplex K of X as the union of the closed simplices whose interior meet $\mathrm{Geod}(D(\xi), x)$. Let v be a vertex of $D(\xi)$. For every simplex σ contained in $(\mathrm{st}(v) \cap K) \setminus D(\xi)$, let $U_{v,\sigma}$ be an open neighbourhood of ξ in $\overline{EG_v}$ disjoint from $\overline{EG_\sigma}$. We then set

$$V_v = \bigcap_{\sigma \subset (\mathrm{st}(v) \cap K) \setminus D(\xi)} U_{v,\sigma}.$$

Now take $\mathcal{U}$ to be a ξ-family contained in $\{V_v, v \in V(\xi)\}$, and let $\mathcal{U}'$ be a ξ-family that is 2-refined in $\mathcal{U}$.

We now show by contradiction that $U' \cap \widetilde{\mathrm{Cone}}_{\mathcal{U}', \frac{1}{2}}(\xi) = \varnothing$. Let y be an point of this intersection. Then $[v_0, y]$ meets $D(\xi)$ (by Corollary IV.4.9) and $B(x, \delta) \cap S(v_0, R)$ (by construction of U').

Since $d(U', X \setminus U) \geq A+1$ and $D(\xi)$ meets $X \setminus U$, it follows that $\mathrm{N}(D(\xi)) \cap U' = \varnothing$. Hence the geodesic segment $[v_0, y]$ enters $D(\xi)$ before meeting $B(x, \delta) \cap S(v_0, R)$. Let y' be the point of $[v_0, y]$ inside $B(x, \delta) \cap S(v_0, R)$. By construction of R and δ, it follows that $\sigma_{y'}$ is in the open star of σ_x. Now since $x \in \mathrm{Cone}_{\mathcal{U}', \frac{1}{2}}(\xi)$, the refinement lemma IV.4.10 implies that $\sigma_{y'} \subset \mathrm{Cone}_{\mathcal{U}, \frac{1}{2}}(\xi)$, which contradicts the definition of $\mathcal{U}$. □

Lemma IV.6.11. *Let $\xi \in \partial_{Stab} G$, $\varepsilon \in (0,1)$ and $\mathcal{U}$ a ξ-family. Then there exists a ξ-family $\mathcal{U}'$ such that every point not in $V_{\mathcal{U},\varepsilon}(\xi)$ admits a neighbourhood that misses $V_{\mathcal{U}',\varepsilon}(\xi)$.*

Proof. Recall that domains of points of $\partial_{Stab}G$ contain at most $d_{\max}$ simplices (see Definition IV.3.3). Choose a ξ-family $\mathcal{U}'$ which is $d_{\max}$-refined and nested in $\mathcal{U}$. We now show that every point not in $V_{\mathcal{U},\varepsilon}(\xi)$ admits a neighbourhood that misses $V_{\mathcal{U}',\varepsilon}(\xi)$.

Let $\widetilde{x} \in EG \setminus V_{\mathcal{U},\varepsilon}(\xi)$, and $x = p(\widetilde{x})$.

- If $x \in \overline{D^\varepsilon(\xi)}$, let v be a vertex of $D(\xi) \cap \sigma_x$. We have $\phi_{v,\sigma_x}(\widetilde{x}) \notin U_v$, hence $\phi_{v,\sigma_x}(\widetilde{x}) \notin \overline{U'_v}$. Let W_x be a neighbourhood of $\phi_{v,\sigma_x}(\widetilde{x})$ in $\overline{EG_v}$ that does not meet U'_v, and V be an open neighbourhood of x in X contained in $\mathrm{st}(\sigma_x)$. Let W be the neighbourhood of $\widetilde{x}$ consisting of those elements $\widetilde{y} \in EG$ whose projection $p(\widetilde{y})$ is in V and such that $\phi_{v,\sigma_x}(\widetilde{y})$ belongs to W_x. Since $\mathcal{U}'$ is refined in $\mathcal{U}$, it then follows that W is a neighbourhood of $\widetilde{x}$ which does not meet $V_{\mathcal{U}',\varepsilon}(\xi)$.

- If $x \notin \overline{D^\varepsilon(\xi)}$, let V be an open neighbourhood of x in $X \setminus D^\varepsilon(\xi)$ contained in $\mathrm{st}(\sigma_x)$. As $\mathcal{U}'$ is refined in $\mathcal{U}$ and $x \notin V_{\mathcal{U},\varepsilon}(\xi)$, Lemma IV.4.10 implies that $p^{-1}(V)$ is a neighbourhood of $\widetilde{x}$ that does not meet $V_{\mathcal{U}',\varepsilon}(\xi)$.

Let $\eta \in \partial X \setminus V_{\mathcal{U},\varepsilon}(\xi)$. We construct a neighbourhood V of η in $\overline{X}$ that does not meet $\widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon}(\xi)$. First, since $D(\xi)$ is bounded, let $R > 0$ such that $D(\xi)$ is contained in the R-ball centred at v_0, and let $x = \gamma_\eta(R+1)$.

- If $[v_0, \eta)$ does not meet $D(\xi)$, let $\delta = \frac{1}{2}\mathrm{dist}\left(\gamma_\eta\big([0,R+1]\big), D(\xi)\right) > 0$, and let V be a neighbourhood of η in $\overline{X}$ that is contained in $V_{R+1,\delta}(\eta)$. For every y in V, $[v_0, y]$ does not meet $D(\xi)$, hence $V \cap \widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon}(\xi) = \varnothing$.

- If $[v_0, \eta)$ goes through $D(\xi)$, then since x does not belong to $\widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$, let v be a vertex of $D(\xi)$ in $\sigma_{\xi,\varepsilon}(x)$ such that $\overline{EG}_{\sigma_{\xi,\varepsilon}}(x) \nsubseteq U_v$ in $\overline{EG_v}$. Lemma IV.4.11 yields a constant $\delta > 0$ such that for every $y \in B(x,\delta)$, $[v_0, y]$ goes through $D^\varepsilon(\xi)$ and $\sigma_{\xi,\varepsilon}(y) \subset \mathrm{st}\big(\sigma_{\xi,\varepsilon}(x)\big)$. Let $V := V_{R+1,\delta}(\eta)$ and $y \in V$. Then $[v_0, y]$ goes through $B(x,\delta)$, hence $\sigma_{\xi,\varepsilon}(y) \subset \mathrm{st}\big(\sigma_{\xi,\varepsilon}(x)\big)$. As $\mathcal{U}'$ is nested in $\mathcal{U}$ and $\overline{EG}_{\sigma_{\xi,\varepsilon}(x)} \nsubseteq U_v$ in $\overline{EG_v}$, it follows that $\overline{EG}_{\sigma_{\xi,\varepsilon}(y)} \nsubseteq U'_v$, hence $y \notin \widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon'}(\xi)$ and $V \cap \widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon'}(\xi) = \varnothing$.

Let $\xi' \in (\partial_{Stab}G) \setminus V_{\mathcal{U},\varepsilon}(\xi)$. To find a neighbourhood of ξ' that misses $V_{\mathcal{U}',\varepsilon}(\xi)$, it is enough, since cones are open subsets of X by Corollary IV.4.12, to find a ξ'-family $\mathcal{U}''$ such that $\widetilde{\mathrm{Cone}}_{\mathcal{U}'',\varepsilon}(\xi') \cap \widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon}(\xi) = \varnothing$ and such that for every vertex v of $D(\xi) \cap D(\xi')$, we have $U'_v \cap U''_v = \varnothing$. We define such a ξ'-family as follows. By Lemma IV.1.5, let K be a finite subcomplex containing $\mathrm{Geod}(v_0, D(\xi))$. Let v be a vertex of $D(\xi')$. For every $\sigma \subset (\mathrm{st}(v) \cap K) \setminus D(\xi')$, let $U''_{v,\sigma}$ be a neighbourhood of ξ' in $\overline{EG_v}$ which is disjoint from $\overline{EG_\sigma}$, and set

$$U''_v = \bigcap_{\sigma \subset (\mathrm{st}(v)\cap K)\setminus D(\xi')} U''_{v,\sigma}.$$

If v is also in $D(\xi)$, note that since the closure of U'_v is contained in U_v, we can assume that $U'_v \cap U''_v = \varnothing$. Furthermore, we can assume by the convergence property IV.3.8 that the only EG_σ inside EG_v meeting both U_v and U''_v contain ξ and ξ'. Now let $\mathcal{U}''$ be a ξ'-family which is $d_{\max}$-refined in $\left\{U''_v, v \in D(\xi')\right\}$.

Let us prove by contradiction that $\widetilde{\mathrm{Cone}}_{\mathcal{U}'',\varepsilon'}(\xi') \cap \widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon}(\xi) = \varnothing$. Let x be in such an intersection. Then, by Corollary IV.4.9, the geodesic $[v_0, x]$ goes through both $D(\xi)$ and $D(\xi')$. Note that, by construction of the various neighbourhoods U''_v, the geodesic segment $[v_0, x]$ cannot leave $D(\xi')$ before leaving $D(\xi)$; nor can it leave both $D(\xi)$ and $D(\xi')$ at the same time. If $D(\xi) \cap D(\xi') = \varnothing$, it follows from the fact that $\mathcal{U}'$ is $d_{\max}$-refined in $\mathcal{U}$ that $D(\xi') \subset \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$ by Lemma IV.4.10, hence $\xi' \in V_{\mathcal{U},\varepsilon}(\xi)$, which is absurd. Otherwise, let x' be the last point of $D(\xi')$ met by $[v_0, x]$ and let γ be a geodesic path in $D(\xi')$ from x' to a point of $D(\xi)$, such that γ meets $D(\xi)$ in exactly one point. Let σ be the last simplex touched by γ before touching $D(\xi)$. The fact that $\mathcal{U}'$ is $d_{\max}$-refined in $\mathcal{U}$ implies that $\overline{EG_\sigma} \subset U_v$ for some (hence every) vertex v of $\sigma \cap D(\xi)$ by Lemma IV.4.10, hence $\xi' \in U_v \subset V_{\mathcal{U},\varepsilon}(\xi)$, a contradiction.

Finally, for every vertex v of $D(\xi) \cap D(\xi')$, we have $U'_v \cap U''_v = \varnothing$ by construction of U''_v, hence the result. □

Theorem IV.6.12. *The space $\overline{EG}$ is separable and metrisable.*

Proof. It is second countable by Theorem IV.5.17, regular by Proposition IV.6.8 and satisfies the T_0 condition by Proposition IV.6.1. Thus it is Hausdorff and the result follows from Urysohn's metrisation theorem. □

IV.6.3 Sequential Compactness.

In this subsection, we prove the following:

Theorem IV.6.13. *The metrisable space $\overline{EG}$ is compact.*

First of all, note that since EG is dense in $\overline{EG}$ by Theorem IV.5.17, it is enough to prove that any sequence in EG admits a subsequence converging in $\overline{EG}$. Let $(\widetilde{x_n})_{n\geq 0} \in (EG)^{\mathbb{N}}$. For every $n \geq 0$, let $x_n = p(\widetilde{x_n})$. Furthermore, to every x_n we associate the finite sequence $\sigma_0^{(n)} = v_0, \sigma_1^{(n)}, \ldots,$ of simplices met by $[v_0, x_n]$. Finally, let $l_n \geq 1$ be the number of simplices of such a sequence.

Lemma IV.6.14. *Suppose that for all $k \geq 0$, $\left\{\sigma_k^{(n)}, n \geq 0\right\}$ is finite.*

- *If (l_n) admits a bounded subsequence, then $(\widetilde{x_n})$ admits a subsequence that converges to a point of $EG \cup \partial_{Stab} G$.*
- *Otherwise, $(\widetilde{x_n})$ admits a subsequence that converges to a point of ∂X.*

Proof. Up to a subsequence, we can assume that there exist open simplices $\sigma_0, \sigma_1, \ldots$ such that for all $k \geq 0, (\sigma_k^{(n)})_{n \geq 0}$ is eventually constant at σ_k. There are two cases to consider:

(i) Up to a subsequence, there exists a constant $m \geq 0$ such that each geodesic $[v_0, x_n]$ meets at most m simplices. This implies that the x_n live in a finite subcomplex. Up to a subsequence, we can now assume that there exists a (closed) simplex σ of X such that x_n is in the interior of σ for all $n \geq 0$. This in turn implies that $\widetilde{x_n}$ is in $\sigma \times \overline{EG_\sigma}$ (or more precisely in the image of $\sigma \times \overline{EG_\sigma}$ in $\overline{EG}$) for all $n \geq 0$. This space is compact since the canonical map $\sigma \times \overline{EG_\sigma} \hookrightarrow \overline{EG}$ is continuous by Lemma IV.5.21, hence we can take a convergent subsequence.

(ii) Up to a subsequence, we can assume that $l_n \to \infty$. For $r > 0$, let $\pi_r : X \to \overline{B}(v_0, r)$ be the retraction on $\overline{B}(v_0, r)$ along geodesics starting at v_0. By assumption, we have that for every $r > 0$, the sequence of projections $(\pi_r(x_n))_{n \geq 0}$ lies in a finite subcomplex of X. A diagonal argument then shows that, up to a subsequence, we can assume that all the sequences of projections $(\pi_m(x_n))_{n \geq 0}$ converge in X for every $m \geq 0$. As the topology of $\overline{X}$ is the topology of the projective limit

$$\overline{B}(v_0, 1) \xleftarrow{\pi_1} \overline{B}(v_0, 2) \xleftarrow{\pi_2} \ldots,$$

it then follows that (x_n) converges in $\overline{X}$. As $l_n \to \infty$, (x_n) converges to a point η of ∂X. The definition of the topology of $\overline{EG}$ now implies that $(\widetilde{x_n})$ converges to η in $\overline{EG}$. □

Lemma IV.6.15. *Suppose that there exists $k \geq 0$ such that $\left\{\sigma_k^{(n)}, n \geq 0\right\}$ is infinite. Then (x_n) admits a subsequence that converges to a point of $\partial_{Stab} G$.*

Proof. Without loss of generality, we can assume that such a k is minimal. Up to a subsequence, we can assume that there exist open simplices $\sigma_1, \ldots, \sigma_{k-1}$ such that for all $n \geq 0$, $\sigma_0^{(n)} = \sigma_0, \ldots, \sigma_{k-1}^{(n)} = \sigma_{k-1}$, and $\left(\sigma_k^{(n)}\right)_{n \geq 0}$ is injective. By cocompactness of the action, we can furthermore assume (up to a subsequence) that the $\sigma_k^{(n)}$ are above a unique simplex of Y. This corresponds to embeddings $\overline{EG}_{\sigma_k^{(n)}} \hookrightarrow \overline{EG}_{\sigma_{k-1}}$. By the convergence property IV.3.8, we can assume, up to a subsequence, that in $\overline{EG}_{\sigma_{k-1}}$ the sequence of subspaces $\overline{EG}_{\sigma_k^{(n)}}$ uniformly converges to a point $\xi \in \partial G_{\sigma_{k-1}}$. Let us prove that $(\widetilde{x_n})_{n \geq 0}$ converges to ξ in $\overline{EG}$.

Since $\overline{EG}$ has a countable basis of neighbourhoods, it is enough to prove that for every $\varepsilon \in (0, 1)$ and every ξ-family $\mathcal{U}$ there exists a subsequence of $(\widetilde{x_n})$ lying in $V_{\mathcal{U}, \varepsilon}(\xi)$. By construction of ξ, we have $\sigma_{k-1} \subset D(\xi)$, and there exists a vertex v_k of $D(\xi)$ such that $\sigma_k^{(n)} \subset \mathrm{st}(v_k)$ for all $n \geq 0$. Two cases may occur:

- Up to a subsequence, all the $[v_0, x_n]$ leave $D^\varepsilon(\xi)$ inside $\sigma_k^{(n)}$. Since $\overline{EG}_{\sigma_k^{(n)}}$ uniformly converges to ξ in $\overline{EG_{\sigma_{k-1}}}$ and thus in $\overline{EG_{v_k}}$, we can assume, up to a subsequence, that $\overline{EG}_{\sigma_k^{(n)}} \subset U_{v_k}$ inside $\overline{EG_{\sigma_k}}$. This implies that $\widetilde{x_n} \in V_{\mathcal{U},\varepsilon}(\xi)$, which is what we wanted.

- Up to a subsequence, all the $[v_0, x_n]$ remain inside $D^\varepsilon(\xi)$ when inside $\sigma_k^{(n)}$. Up to a subsequence, we can further assume that all the $\sigma_{k+1}^{(n)}, n \geq 0$ are above a unique simplex of Y. Thus there exists a vertex v_{k+1} of $D(\xi) \cap \overline{\text{st}}(v_k)$ such that $\sigma_{k+1}^{(n)} \subset st(v_{k+1})$ for all $n \geq 0$.

  In particular we have $\sigma_k^{(n)} \subset \text{st}(v_k) \cap \text{st}(v_{k+1})$ and thus $\xi \in \partial G_{v_{k+1}}$. Since $\mathcal{U}$ is a ξ-family, the fact that $\overline{EG}_{\sigma_k^{(n)}}$ uniformly converges to ξ in $\overline{EG_{v_k}}$ implies that that $\overline{EG}_{\sigma_k^{(n)}}$ uniformly converges to ξ in $\overline{EG_{v_{k+1}}}$. Note that since the sequence $(\sigma_k^{(n)})_{n\geq 0}$ takes infinitely many values, the finiteness lemma IV.1.5 implies that $(\sigma_{k+1}^{(n)})_{n\geq 0}$ also takes infinitely many values. Up to a subsequence, we can thus assume by the convergence property IV.3.8 that $\overline{EG}_{\sigma_{k+1}^{(n)}}$ uniformly converges in $\overline{EG_{v_{k+1}}}$. As $\overline{EG}_{\sigma_k^{(n)}}$ uniformly converges to ξ in $\overline{EG_{v_{k+1}}}$, the same holds for $\overline{EG}_{\sigma_{k+1}^{(n)}}$, and we are back to the previous situation.

By iterating this algorithm, two cases may occur:

- There is a value $k' \geq k$ such that, up to a subsequence, all the $[v_0, x_n]$ leave $D^\varepsilon(\xi)$ while being inside $\sigma_{k'}^{(n)}$ and the same argument as before shows that we can take a subsequence satisfying $\widetilde{x_n} \in V_{\mathcal{U},\varepsilon}(\xi)$.

- Up to a subsequence, at every stage $k' \geq k$ all the $[v_0, x_n]$ remain within $D^\epsilon(\xi)$. In the latter case, the containment lemma IV.1.3 implies that there exists an integer $m \geq 0$ such that each geodesic segment $[v_0, x_n]$ meets at most m simplices. Up to a subsequence, we can further assume that all the $[v_0, x_n]$ meet exactly m simplices. Thus we can iterate our algorithm up to rank m, which yields the existence of a vertex v_m of $D^\varepsilon(\xi)$ such that $\sigma_m^{(n)} \subset \text{st}(v_m)$ for all $n \geq 0$ and such that $\overline{EG}_{\sigma_m^{(n)}}$ uniformly converges to ξ in $\overline{EG_{v_m}}$. Up to a subsequence, we can furthermore assume that $\overline{EG}_{\sigma_m^{(n)}} \subset U_m$ in $\overline{EG}_{v_{k+1}}$ for all $n \geq 0$. This in turn implies $\widetilde{x_n} \in W_{\mathcal{U},\varepsilon}(\xi)$, hence $\widetilde{x_n} \in V_{\mathcal{U},\varepsilon}(\xi)$ and we are done. □

Proof of Theorem IV.6.13. This follows immediately from Theorem IV.6.12, Lemma IV.6.14 and Lemma IV.6.15. □

As a direct consequence, we get the following convergence criterion.

Corollary IV.6.16. *Let (K_n) be a sequence of subsets of $\overline{EG}$.*

- *K_n uniformly converges to a point $\eta \in \partial X$ if and only if the sequence of coarse projections $\bar{p}(K_n)$ uniformly converges to η in $\overline{X}$.*
- *Suppose that there exists $\xi \in \partial_{Stab} G$ such that, for n large enough, every geodesic from v_0 to a point of $\bar{p}(K_n)$ goes through $D(\xi)$. For every such n and every $z \in K_n$, choose $x \in \bar{p}(z)$ and let $\sigma_{n,x}$ be the first simplex touched by the geodesic $[v_0, x]$ after leaving $D(\xi)$. If there exists a vertex $v \in D(\xi)$ contained in each $\sigma_{n,x}$ and such that for every neighbourhood U of ξ in $\overline{EG_v}$, there exists an integer $N \geq 0$ such that for every $(n, x) \in \cup_{n \geq N}\{n\} \times K_n$, we have $\overline{EG}_{\sigma_{n,x}} \subset U$, then (K_n) uniformly converges to ξ.* □

IV.7 The properties of ∂G.

In this section we prove the following:

Theorem IV.7.1. *$(\overline{EG}, \partial G)$ is an $E\mathcal{Z}$-structure.*

IV.7.1 The $\mathcal{Z}$-set property

Here we prove the following:

Proposition IV.7.2. *∂G is a $\mathcal{Z}$-set in $\overline{EG}$.*

Proving this property is generally technical. However, Bestvina and Mess proved in [4] a useful lemma ensuring that a given set is a $\mathcal{Z}$-set in a bigger set, which we now recall.

Lemma IV.7.3 (Bestvina-Mess [4])**.** *Let $(\widetilde{X}, Z)$ be a pair of finite-dimensional metrisable compact spaces with Z nowhere dense in $\widetilde{X}$, and such that $X = \widetilde{X} \setminus Z$ is contractible and locally contractible, with the following condition holding:*

(*) For every $z \in Z$ and every neighbourhood $\widetilde{U}$ of z in $\widetilde{X}$, there exists a neighbourhood $\widetilde{V}$ contained in $\widetilde{U}$ and such that

$$\widetilde{V} \setminus Z \hookrightarrow \widetilde{U} \setminus Z$$

is null-homotopic.

Then $\widetilde{X}$ is an Euclidian retract and Z is a $\mathcal{Z}$-set in $\widetilde{X}$.

We now use this lemma to prove that the boundary ∂G is a $\mathcal{Z}$-boundary in $\overline{EG}$.

Lemma IV.7.4. *$\overline{EG}$ and ∂G are finite-dimensional.*

Proof. We have

$$\partial G = \Bigg(\bigcup_{v \in V(X)} \partial G_v \Bigg) \cup \partial X.$$

Each vertex stabiliser boundary is a $\mathcal{Z}$-boundary in the sense of Bestvina, hence finite-dimensional, and they are closed subspaces of ∂G by Proposition IV.5.19. As the action of G on X is cocompact, their dimension is uniformly bounded above, so the countable union theorem implies that $\bigcup_{v \in V(X)} \partial G_v$ is finite-dimensional. Furthermore, X is a CAT(0) space of finite geometric dimension, so its boundary has finite dimension by a result of Caprace [10]. Thus, the classical union theorem implies that ∂G is finite-dimensional. Now $\overline{EG} = EG \cup \partial G$. EG is a CW-complex that can be decomposed as the countable union of its closed cells, all of which have a dimension bounded above by $\dim(X) \cdot \sup_\sigma(\dim EG_\sigma)$. It follows from the countable union theorem in covering dimension theory that EG is finite dimensional, and the same holds for $\overline{EG}$ by the classical union theorem. □

We now turn to the proof of the $\mathcal{Z}$-set property, using the lemma of Bestvina-Mess recalled above. As usual, the proof splits in two cases, depending on the nature of the point of ∂G that we consider.

Lemma IV.7.5. *Let $\eta \in \partial X$ and U be a neighbourhood of η in $\overline{X}$. Then there exists a subneighbourhood $U'' \subset U$ of η in $\overline{X}$ such that the inclusion*

$$V_{U''}(\eta) \setminus \partial G \hookrightarrow V_U(\eta) \setminus \partial G$$

is null-homotopic.

Proof. Lemma IV.5.3 yields a neighbourhood U' of η in $\overline{X}$ such that $d(U' \cap X, X \setminus U) > 1$. In particular, $\mathrm{Span}(U' \setminus \partial X) \subset U$, and $p^{-1}(\mathrm{Span}(U' \setminus \partial X))$ can be seen as the realisation of a complex of spaces over $\mathrm{Span}(U' \setminus \partial X)$ the fibres of which are contractible. Thus Proposition II.1.8 implies that the projection $p^{-1}(\mathrm{Span}(U' \setminus \partial X)) \to \mathrm{Span}(U' \setminus \partial X)$ is a homotopy equivalence. Now Lemma IV.5.2 yields another neighbourhood $U'' \subset U'$ of η in $\overline{X}$ such that $U'' \setminus \partial X$ is contractible. We thus have the following commutative diagram:

$$\begin{array}{ccccc} V_U(\eta) \setminus \partial G & \hookleftarrow & p^{-1}(\mathrm{Span}(U' \setminus \partial X)) & \hookleftarrow & V_{U''}(\eta) \setminus \partial G \\ & & \downarrow \simeq & & \downarrow \\ & & \mathrm{Span}(U' \setminus \partial X) & \underset{0}{\hookleftarrow} & U'' \setminus \partial X. \end{array}$$

Now since $U'' \setminus \partial X$ is contractible, the inclusion $V_{U''}(\eta) \setminus \partial G \hookrightarrow V_U(\eta) \setminus \partial G$ is null-homotopic. □

Lemma IV.7.6. *Let $\xi \in \partial_{Stab} G$, $\varepsilon \in (0,1)$ and $\mathcal{U}$ a ξ-family. Then there exists a ξ-family $\mathcal{U}'$ such that $V_{\mathcal{U}',\varepsilon}(\xi)$ is a subneighbourhood of $V_{\mathcal{U},\varepsilon}(\xi)$ and such that the inclusion*

$$V_{\mathcal{U}',\varepsilon}(\xi) \setminus \partial G \hookrightarrow V_{\mathcal{U},\varepsilon}(\xi) \setminus \partial G$$

is null-homotopic.

Lemma IV.7.7. *There exists a ξ-family $\mathcal{U}''$, a subcomplex X' of X with $\widetilde{Cone}_{\mathcal{U}'',\varepsilon}(\xi) \subset X' \subset \widetilde{Cone}_{\mathcal{U},\varepsilon}(\xi)$, and a subset C' of EG with $V_{\mathcal{U}'',\varepsilon}(\xi) \setminus \partial G \subset C' \subset V_{\mathcal{U},\varepsilon}(\xi) \setminus \partial G$, such that $p(C') \subset X'$ and the projection map $C' \to X'$ is a homotopy equivalence.*

Proof. Let $\mathcal{U}'$ be a ξ-family that is 2-refined in $\mathcal{U}$ and d_ξ-nested in $\mathcal{U}$. It follows from the refinement lemma IV.4.10 that $\mathrm{Span}\Big(\mathrm{Cone}_{\mathcal{U}',\varepsilon}(\xi)\Big) \subset \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$. By Lemma IV.5.6, we have $V_{\mathcal{U}',\varepsilon}(\xi) \subset V_{\mathcal{U},\varepsilon}(\xi)$. Let

$$X' = \mathrm{Span}(\mathrm{Cone}_{\mathcal{U}',\varepsilon}(\xi)) \cup \Big(\overline{D^\varepsilon(\xi)} \cap \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)\Big).$$

Note that it is possible to give $\overline{D^\varepsilon(\xi)} \cap \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$ a simplicial structure from that of X such that a vertex of $\overline{D^\varepsilon(\xi)} \cap \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$ for that structure either is a vertex of $D(\xi)$ or belongs to an edge in X between a vertex of $D(\xi)$ and a vertex of $X \setminus D(\xi)$. Furthermore, we can give $\mathrm{Span}(\mathrm{Cone}_{\mathcal{U}',\varepsilon}(\xi))$ a simplicial structure that is finer that that of X, whose vertices are the vertices of $\mathrm{Span}(\mathrm{Cone}_{\mathcal{U}',\varepsilon}(\xi))$ and vertices of $\overline{D^\varepsilon(\xi)} \cap \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$ (for its given simplicial structure), that is compatible with that of $\overline{D^\varepsilon(\xi)}$, and which turns X' into a simplicial complex such that an open simplex is completely contained either in $\overline{D^\varepsilon(\xi)}$ or in $X \setminus D^\varepsilon(\xi)$ (see Figure *IV*.5). Thus X' is endowed with a simplicial structure.

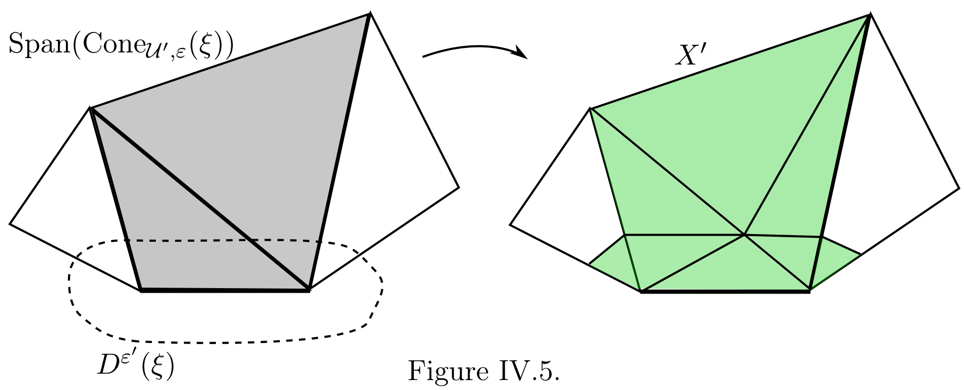

Figure IV.5.

We now define a contractible open subset C'_σ of EG_σ for every open simplex σ of X'. This will allow us to define the following subset of EG:

$$C' = \bigcup_{\sigma \in S(X')} \sigma \times C'_\sigma.$$

Note that although C' is not naturally the realisation of a complex of spaces in the sense of the first section, it is nonetheless possible to endow it with one, so as to use Proposition II.1.8.

We first define these spaces C'_σ for vertices of X'. Let v be such a vertex.

- If v is a vertex of $D(\xi)$, the compactification $\overline{EG_v}$ is locally contractible so we can choose a contractible open set U'_v of $\overline{EG_v}$ contained in $\overline{U_v}$ and containing ξ, and set $C'_v = U'_v \cap EG_v$. As ∂G_v is a $\mathcal{Z}$-boundary, C'_v is a contractible open subset.

- If v does not belong to $D^\varepsilon(\xi)$, set $C'_v = \overline{EG_v}$.

- If v is a vertex of $\overline{D^\varepsilon(\xi)} \setminus D(\xi)$ (for the chosen simplicial structure of $\overline{D^\varepsilon(\xi)} \subset X'$), then either v belongs to $\mathrm{Span}(\mathrm{Cone}_{\mathcal{U}',\varepsilon}(\xi))$, in which case we set $C'_v = EG_v$, or it does not, in which case v belongs to a unique edge e (for the simplicial structure of X) between a vertex v' of $D(\xi)$ and a vertex of $X \setminus D(\xi)$. In that case, $\overline{EG_e}$ is contained in $U_{v'}$ since $\mathcal{U}'$ is nested in $\mathcal{U}$ and we set $C'_v = EG_e$.

We now define the subsets C'_σ for simplices $\sigma \subset X'$. Let σ be such a simplex, and let σ' be the unique open simplex of X such that $\sigma \subset \sigma'$ as subsets of X. We set $C'_\sigma = EG_{\sigma'}$.

We define the space $C' = \bigcup_{\sigma \in S(X')} \sigma \times C'_\sigma$. As explained above, the projection $C' \to X'$ is a homotopy equivalence. Furthermore, we can choose a ξ-family $\mathcal{U}''$ small enough so that the subset $V_{\mathcal{U}'',\varepsilon}(\xi) \setminus \partial G$ is contained in C'. □

Proof of Lemma IV.7.6. We apply the previous lemma twice to get the following commutative diagram:

$$\begin{array}{ccccccc}
V_{\mathcal{U},\varepsilon}(\xi) \setminus \partial G & \hookleftarrow & C' & \hookleftarrow & V_{\mathcal{U}'',\varepsilon}(\xi) \setminus \partial G & \hookleftarrow & C^{(3)} \\
 & & \downarrow \simeq & & \downarrow & & \downarrow \simeq \\
 & & X' & \hookleftarrow & \widetilde{\mathrm{Cone}}_{\mathcal{U}'',\varepsilon}(\xi) & \overset{}{\underset{0}{\hookleftarrow}} & X^{(3)}.
\end{array}$$

Since $X^{(3)}$ retracts by strong deformation (along geodesics starting at v_0) inside $\widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon}(\xi)$ on the contractible subcomplex $D(\xi)$ (relatively to $D(\xi)$), the inclusion $X^{(3)} \hookrightarrow \widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon}(\xi)$ is nullhomotopic, hence $C^{(3)} \hookrightarrow V_{\mathcal{U},\varepsilon}(\xi) \setminus \partial G$ is null-homotopic. As there exists a ξ-family $\mathcal{U}^{(4)}$ such that $V_{\mathcal{U}^{(4)},\varepsilon}(\xi) \setminus \partial G \hookrightarrow C^{(3)}$, this concludes the proof. □

Proof of Proposition IV.7.2: Thus, Theorem IV.6.13 and Lemma IV.7.4 together with Lemma IV.7.5 and Lemma IV.7.6 yield the desired result. □

IV.7.2 Compact sets fade at infinity

Here we prove the following:

Proposition IV.7.8. *Compacts subsets of EG fade at infinity in $\overline{EG}$, that is, for every $x \in \partial G$, every neighbourhood U of x in $\overline{EG}$ and every compact $K \subset EG$, there exists a subneighbourhood $V \subset U$ of x such that any G-translate of K meeting V is contained in U.*

As usual, we split the proof in two parts, depending on the nature of the points considered.

Proposition IV.7.9. *Let $\eta \in \partial X$. For every neighbourhood U of η in $\overline{X}$ and every compact subset $K \subset EG$, there exists a neighbourhood U' of η contained in U and such that any G-translate of K meeting $V_{U'}(\eta)$ is contained in $V_U(\eta)$.*

Proof. By Lemma IV.5.3, let U' be a neighbourhood of η in $\overline{X}$ which is contained in U and such that

$$d(U', X \setminus U) > \mathrm{diam}(p(K)).$$

Let $g \in G$ such that gK meets $V_{U'}(\eta)$. Since G acts on X by isometries, we have

$$\mathrm{diam}\,(p(g.K)) = \mathrm{diam}\,(g.p(K)) = \mathrm{diam}\,(p(K))\,,$$

which implies that $gK \subset V_U(\eta)$. □

The proof for points of $\partial_{Stab} G$ is slightly more technical. We start by defining a class of compact sets of EG which are easy to handle.

Definition IV.7.10. Let F be a finite subcomplex of X, together with a collection $(K_\sigma)_{\sigma \in S(F)}$ of non empty compact subsets of EG_σ for every simplex σ of F. Suppose that for every simplex σ of F and every face σ' of σ, we have $\phi_{\sigma',\sigma}(K_\sigma) \subset K_{\sigma'}$. Then the set

$$\bigcup_{\sigma \in \mathcal{S}(F)} \sigma \times K_\sigma.$$

is called a *standard compact subset of EG over F*. Every compact subset of EG obtained in such a way is called a standard compact of EG.

Note that the projection in X of any compact subset of EG meets finitely many simplices of X, so every compact subset of EG may be seen as a subset of a standard compact subset of EG.

Definition IV.7.11. Let $\xi \in \partial_{Stab} G$ and $\mathcal{U}$ a ξ-family. We define $W_{\mathcal{U}}(\xi)$ as the set of points $\widetilde{x}$ of EG whose projection $x \in X$ belongs to the domain of ξ and is such that for some (hence any) vertex v of $\sigma_x \cap D(\xi)$ we have

$$\phi_{v,\sigma_x}(\widetilde{x}) \in U_v.$$

Before proving that compact sets fade near points of $\partial_{Stab} G$, we prove the following lemma.

Lemma IV.7.12. *Let $\xi \in \partial_{Stab} G$, $\varepsilon \in (0,1)$ and $\mathcal{U}$ a ξ-family. Let K be a compact subspace of EG. Then there exists a ξ-family $\mathcal{U}'$ contained in $\mathcal{U}$ such that for every point $g \in G$, the following holds:*

If gK meets $W_{\mathcal{U}'}(\xi)$, then $gK \cap p^{-1}(D(\xi))$ is contained in $W_{\mathcal{U}}(\xi)$.

Proof. Let L be a standard compact subset of EG over the (finite) full subcomplex of X defined by Span $p(K)$. By choosing the L_σ big enough, we can assume that L contains K. Let $N \geq 0$ be such that any two vertices of L can be joined by a sequence of at most N adjacent vertices.

Since $D(\xi)$ and $p(L)$ meet finitely many vertices of X, there are only finitely many elements of G such that $g.p(L)$ meets $D(\xi)$ up to left multiplication by an element of $G_v, v \in V(\xi)$. Let $(g_\lambda.p(L))_{\lambda\in\Lambda}$ be such a finite family of cosets. For every vertex v of $V(\xi)$, $\{g_\lambda L \cap EG_v, \lambda \in \Lambda\}$ is a finite (possibly empty) collection of compact subsets of EG_v. Since ∂G_v is a Bestvina boundary for G_v, compact subsets fade at infinity in $\overline{EG_v}$, so there exists a subneighbourhood U'_v of U_v such that any G_v-translate of one of these $g_\lambda L$ meeting U'_v is contained in U_v. Repeating this procedure $N+1$ times, we get a sequence of ξ-families denoted

$$\{U_v, v \in V(\xi)\} \supset \mathcal{U}^{[N]} \supset \mathcal{U}^{[N-1]} \supset \ldots \supset \mathcal{U}^{[0]}.$$

Let $g \in G$ such that gK meets $W_{\mathcal{U}'}(\xi)$, and let w be a vertex of $D(\xi)$ such that gK, hence gL, meets $U_w^{[0]}$. In order to prove the lemma, it is enough to show by induction on $k = 0, \ldots, N$ the following:

(H_k) : For every chain of adjacent vertices $w_0 = w, w_1, \ldots, w_k$ of $D(\xi)$ such that gL meets $EG_{w_0}, \ldots, EG_{w_k}$, we have $gL \cap EG_{w_k} \subset U_w^{[k+1]}$.

Since gL meets $D(\xi)$, let $\lambda \in \Lambda$ such that $gL = g_\lambda L$ pointwise. The result is true for $k = 0$ by definition of $\mathcal{U}^{[0]}$ and $\mathcal{U}^{[1]}$. Suppose we have proven it up to rank k, and let $w_0 = w, w_1, \ldots, w_{k+1}$ a chain of vertices of $D(\xi)$ such that gL meets $EG_{w_0}, \ldots, EG_{w_k}$. By induction hypothesis, we already have $gL \cap EG_{w_k} \subset U_{w_k}^{[k+1]}$. Since $p(L)$ is a full subcomplex of X, it follows from the fact that gL meets EG_{w_k} and $EG_{w_{k+1}}$ that gL also meets $EG_{[w_k,w_{k+1}]}$. In particular, since $gL \cap EG_{w_k} \subset U_{w_k}^{[k+1]}$, it follows from the properties of ξ-families that $gL \cap EG_{w_{k+1}}$ meets $U_{w_{k+1}}^{[k+1]}$. This in turn implies that $gL \cap EG_{w_{k+1}} \subset U_{v_{k+1}}^{[k+2]}$, which concludes the induction. □

Proposition IV.7.13. *Let $\xi \in \partial_{Stab}G$, $\varepsilon \in (0,1)$ and $\mathcal{U}$ a ξ-family. Let K be a connected compact subset of EG. Then there exists a ξ-family $\mathcal{U}'$ contained in $\mathcal{U}$ and such that every G-translate of K meeting $V_{\mathcal{U}',\varepsilon}(\xi)$ is contained in $V_{\mathcal{U},\varepsilon}(\xi)$.*

Proof. Let k be the number of simplices met by $p(K)$, and let $\mathcal{U}'$ be a ξ-family that is k-refined in $\mathcal{U}$. Applying the previous proposition to $V_{\mathcal{U}',\varepsilon}(\xi)$ yields a ξ-family $\mathcal{U}''$. Finally, let $\mathcal{U}'''$ be a ξ-family that is k-refined in $\mathcal{U}''$.

Suppose that gK meets $V_{\mathcal{U}''',\varepsilon}(\xi)$, and let $\widetilde{x_0} \in gK \cap V_{\mathcal{U}''',\varepsilon}(\xi)$. Let $\widetilde{x} \in gK$, and let us prove that $\widetilde{x} \in V_{\mathcal{U},\varepsilon}(\xi)$. Since $p(K)$ is connected, let γ be a path from $x_0 = p(\widetilde{x_0})$ to $x = p(\widetilde{x})$ in $p(gK)$. This yields a path of open simplices $\sigma_1, \ldots, \sigma_n$, with $n \leq k$. If

gK does not meet $D(\xi)$, the refinement lemma IV.4.10 implies that $\sigma_n \subset \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi)$, and $\widetilde{x} \in V_{\mathcal{U},\varepsilon}(\xi)$.

Otherwise, let n_0 (resp. n_1) be such that σ_{n_0} (resp. σ_{n_1}) is the first (resp. the last) simplex contained in $D(\xi)$. If x_0 is not in $D(\xi)$, we can apply the refinement lemma IV.4.10 to the path $\sigma_1, \ldots, \sigma_{n_0-1}$, which implies $\sigma_{n_0-1} \subset \mathrm{N}_{\mathcal{U}''}(D(\xi))$. In particular, we see that gK meets $W_{\mathcal{U}''}(\xi)$, which is also true if x_0 is in $D(\xi)$. Now by definition of $\mathcal{U}''$, we have that $gK \cap p^{-1}(D(\xi)) \subset W_{\mathcal{U}'}(\xi)$. If γ goes out of $D(\xi)$ after σ_{n_1}, then $\sigma_{n_1+1} \subset \mathrm{N}_{\mathcal{U}'}(D(\xi))$, and we can apply the refinement lemma IV.4.10 to the path of simplices $\sigma_{n_1+1}, \ldots, \sigma_n$. In any case, we get in the end $\widetilde{x} \in V_{\mathcal{U},\varepsilon}(\xi)$, which concludes the proof. □

Proof of Proposition IV.7.8: This follows from Proposition IV.7.9 and Proposition IV.7.13. □

Proof of Theorem IV.7.1: This follows from Theorem IV.6.13, Lemma IV.5.18, Proposition IV.7.2, and Proposition IV.7.8. □

IV.7.3 Proof of the main theorem.

We are now ready to conclude the proof of Theorem IV.0.4.

Lemma IV.7.14. *Let X, Y and G as in the statement of the main theorem. Then for every simplex σ of Y, the embedding $\overline{EG_\sigma} \hookrightarrow \overline{EG}$ realises an equivariant homeomorphism from ∂G_σ to $\Lambda G_\sigma \subset \partial G$. Moreover, for every pair H_1, H_2 of subgroups in the family $\mathcal{F} = \left\{\bigcap_{i=1}^n g_i G_{\sigma_i} g_i^{-1} \mid g_1, \ldots, g_n \in G,\ \sigma_1, \ldots, \sigma_n \in S(Y),\ n \in \mathbb{N}\right\}$, we have $\Lambda H_1 \cap \Lambda H_2 = \Lambda(H_1 \cap H_2) \subset \partial G$.*

Proof. The equivariant embedding $\overline{EG_\sigma} \hookrightarrow \overline{EG}$ induces an equivariant embedding $\partial G_\sigma \hookrightarrow \Lambda G_\sigma \subset \partial G$. But since $\overline{EG_\sigma}$ is a closed subspace of $\overline{EG}$ by Proposition IV.5.19, and which is stable under the action of G_σ, the reverse inclusion $\Lambda G_\sigma \subset \partial G_\sigma$ follows.

Now let $\sigma_1, \ldots, \sigma_n$ be simplices of X. The inclusion

$$\Lambda\Big(\bigcap_{1 \leq i \leq n} G_{\sigma_i}\Big) \subset \bigcap_{1 \leq i \leq n} \Lambda G_{\sigma_i}$$

is clear, and the reverse inclusion follows directly from Lemma IV.3.7. □

Lemma IV.7.15. *Let X and G be as in the statement of the main theorem. Then for every simplex σ of X, the embedding $\overline{EG_\sigma} \hookrightarrow \overline{EG}$ satisfies the convergence property IV.3.8.*

Proof. Let $(g_n G_\sigma)$ be a sequence of distinct G-cosets. This yields an injective sequence of simplices $(g_n\sigma)$ of X. Let $\widetilde{x}$ be any point of EG_σ. By compactness of $\overline{EG}$, we can assume up to a subsequence that $g_n\widetilde{x}$ converges to a point $l \in \overline{EG}$. But it follows immediately from Lemma IV.6.14 and Lemma IV.6.15 that $l \in \partial G$ and that $g_n\overline{EG_\sigma}$ uniformly converges to l. □

Lemma IV.7.16. *Let X and G be as in the statement of the main theorem. Then for every simplex σ of X, the group G_σ is of finite height in G.*

Proof. Let $g_1G_\sigma, \dots, g_nG_\sigma$ be distinct G-cosets such that $g_1G_\sigma g_1^{-1} \cap \dots \cap g_nG_\sigma g_n^{-1}$ is infinite. Thus the simplices $g_1\sigma, \dots, g_n\sigma$ of X are distinct and such that the boundary of their stabilisers have a nonempty intersection in $\partial_{Stab}G$. But as there is a uniform bound on the number of simplices contained in the domain of a point of $\partial_{Stab}G$ by Proposition IV.3.2, Lemma IV.3.6 implies that there is a uniform bound on the number of simplices whose stabilisers have an infinite intersection, hence the result. □

Proof of Theorem IV.0.4: This follows from Theorem IV.7.1, Lemma IV.7.14, Lemma IV.7.15 and Lemma IV.7.16. □

IV.7.4 Boundaries in the sense of Carlsson-Pedersen.

So far we have been concerned with the notion of an $E\mathcal{Z}$-structure. We now turn to the notion of an $E\mathcal{Z}$-structure in the sense of Carlsson-Pedersen. In order to obtain a combination theorem for such finer structures, we need an additional assumption that we now describe.

Definition IV.7.17. We say that an $E\mathcal{Z}$-structure in the sense of Carlsson-Pedersen $(\overline{EG}, \partial G)$ is *strong* if in addition we have the following:

For every finite group H of G, $(\partial G)^H$ is either empty or a $\mathcal{Z}$-set in $\overline{EG}^H$.

Without any assumption of a strong $E\mathcal{Z}$-structure, it is still possible to prove the following partial result.

Lemma IV.7.18. *Let $H \subset G$ be a finite subgroup. Then the closure of EG^H in $\overline{EG}$ is exactly $\overline{EG}^H$.*

Proof. As EG is a classifying space for proper actions of G, EG^H is nonempty. We now prove that it is dense in $\overline{EG}^H$.

Let $\xi \in \partial_{Stab}G \cap \overline{EG}^H$. The domain $D(\xi)$ is thus stable under the action of H. As $D(\xi)$ is a finite convex subcomplex of X, the fixed point theorem for CAT(0) spaces implies that there is a point of $D(\xi)$ fixed by H. Since the action is without inversion, we can further assume that H fixes a vertex v of $D(\xi)$. Moreover, EG_v^H is dense in $\overline{EG_v^H}$. Thus, by definition of a basis of neighbourhoods at ξ, any neighbourhood of ξ in $\overline{EG}$ meets EG^H.

Now let $\eta \in \partial X \cap \overline{EG}^H$. Let γ be a geodesic from a point of X^H to η. Then γ is fixed pointwise by H. Let U be a neighbourhood of η in $\overline{X}$. Since the path γ eventually meets U, let σ be a simplex of X contained in U and met by γ. Thus σ is fixed pointwise by H. Now since EG_σ^H is nonempty by assumption, it follows that EG^H meets $V_U(\eta)$, and the result follows. □

However, the previous reasoning does not show the contractibility of $\overline{EG}^H$. We now reformulate our main theorem in the setting of $E\mathcal{Z}$-structures in the sense of Carlsson-Pedersen.

Definition IV.7.19. An $E\mathcal{Z}$-complex of classifying spaces *in the sense of Carlsson-Pedersen* (compatible with the complex of groups $G(\mathcal{Y})$) is an $E\mathcal{Z}$-complex of classifying spaces compatible with $G(\mathcal{Y})$ such that each local $E\mathcal{Z}$-structure $(\overline{EG_\sigma}, \partial G_\sigma)$ is a strong $E\mathcal{Z}$-structure in the sense of Carlsson-Pedersen.

Theorem IV.7.20. *The combination theorem for boundaries of groups IV.0.4 remains true if ones replaces "$E\mathcal{Z}$-complexes of classifying spaces" with "$E\mathcal{Z}$-complexes of classifying spaces in the sense of Carlsson-Pedersen".*

Proof. The only thing to prove is that $(\overline{EG}, \partial G)$ is an $E\mathcal{Z}$-structure in the sense of Carlsson-Pedersen. We already know that it is an $E\mathcal{Z}$-structure by Theorem IV.0.4. Let H be a finite subgroup of G. To prove that $\overline{EG}^H$ is contractible, we want to apply the lemma IV.7.3 of Bestvina-Mess to the pair $(\overline{EG}^H, \overline{EG}^H \setminus EG^H)$.

In order to do this, first notice that EG^H is nothing but the complex of spaces over X^H with fibres the subcomplexes EG_σ^H of EG_σ. Thus, it is possible to apply the exact same reasoning with X^H in place of X and the EG_σ^H in place of the EG_σ. As X^H is a convex, hence contractible subcomplex of X, this is enough to recover the fact that EG^H is contractible.

Now, notice that, because of Lemma IV.7.18, $\overline{EG}^H$ is obtained from EG^H by the same procedure as before, compactifying every EG_σ^H (for σ a simplex fixed under H) by $\overline{EG_\sigma}^H$ and adding the visual boundary of the CAT(0) subcomplex X^H, $\partial(X^H) = (\partial X)^H$. We now briefly indicate why this is enough to prove the $\mathcal{Z}$-set property for $(\overline{EG}^H, \overline{EG}^H \setminus EG^H)$. The only properties that were required are the fact that X is a CAT(0) space, the convergence properties of the embeddings between the various classifying spaces, and the fact that ∂G_σ is a $\mathcal{Z}$-set in $\overline{EG_\sigma}$. But since X^H is convex in a CAT(0) space, it is itself CAT(0). Moreover, the convergence properties of the embeddings are clearly still satisfied for simplices that are fixed under H. Finally, by assumption, $(\partial G_\sigma)^H$ is a $\mathcal{Z}$-set in $\overline{EG_\sigma}^H$. Thus, the same reasoning as in Lemma IV.7.5 and Lemma IV.7.6 shows that the lemma IV.7.3 of Bestvina-Mess applies, thus implying that $(\overline{EG}^H, \overline{EG}^H \setminus EG^H)$ is a $\mathcal{Z}$-compactification, and we are done. □

Chapter V

A combination theorem for hyperbolic groups.

In this chapter, we apply our construction of boundaries to get a generalisation of a combination theorem of Bestvina-Feighn to complexes of groups of arbitrary dimension. This will be done by constructing an $E\mathcal{Z}$-structure for G and proving that G is a uniform convergence group on its boundary. The proof has the advantage of yielding a construction of the Gromov boundary of G.

Theorem V.0.21 (Combination Theorem for Hyperbolic Groups)**.** *Let $G(\mathcal{Y})$ be a non-positively curved complex of groups over a finite simplicial complex Y endowed with a M_κ-structure, $\kappa \leq 0$. Let G be the fundamental group of $G(\mathcal{Y})$ and X be a universal covering of $G(\mathcal{Y})$. Assume that:*

- *The universal covering X is hyperbolic*[1]*.*
- *The local groups are hyperbolic and all the local maps are quasiconvex embeddings,*
- *The action of G on X is acylindrical.*

Further assume that there exists an $E\mathcal{Z}$-complex of classifing spaces compatible with $G(\mathcal{Y})$. Then G is hyperbolic and the local groups embed in G as quasiconvex subgroups.

An important case where compatible $E\mathcal{Z}$-complexes of classifying spaces always exist is the case of simple complexes of hyperbolic groups (see Lemma V.1.1). We thus get the following:

Corollary V.0.22. *Let $G(\mathcal{Y})$ be a simple non-positively curved complex of groups over a finite simplicial complex Y endowed with a M_κ-structure, $\kappa \leq 0$. Let G be the fundamental group of $G(\mathcal{Y})$ and X be a universal covering of $G(\mathcal{Y})$. Assume that:*

[1] For instance, when $\kappa < 0$.

- *The universal covering X is hyperbolic.*
- *The local groups are hyperbolic and all the local maps are quasiconvex embeddings,*
- *The action of G on X is acylindrical.*

Then G is hyperbolic and the local groups embed in G as quasiconvex subgroups.

V.1 Complexes of hyperbolic groups and their $E\mathcal{Z}$-complexes of classifying spaces.

Lemma V.1.1. *Let $G(\mathcal{Y})$ be a simple complex of hyperbolic groups. Then there exists an $E\mathcal{Z}$-complex of classifying spaces compatible with $G(\mathcal{Y})$.*

Proof. We define an $E\mathcal{Z}$-complex of classifying spaces over Y as follows:

- We define inductively sets of generators for the local groups of the complex of groups $G(\mathcal{Y})$ induced over Y in the following way: Start with simplices σ of Y of maximal dimension, and choose for each of them a finite symmetric set of generators for G_σ. Suppose we have defined a set of generators for local groups over simplices of dimension at most k. If σ is a simplex of dimension $k-1$, choose a finite set of generators which contains all the generators of local groups of simplices strictly containing σ. This allows us to define for every simplex σ of Y a set of generator such that $\psi_{\sigma,\sigma'}(S_{\sigma'}) \subset S_\sigma$ whenever $\sigma \subset \sigma'$.
- Let $n \geq 1$ be an integer. Define D_σ as the Rips complex $P_n(G_\sigma)$ associated to the set of generators S_σ. Moreover, if $\sigma \subset \sigma'$, let $\phi_{\sigma,\sigma'}$ be the $\psi_{\sigma,\sigma'}$-equivariant embedding $P_n(G_{\sigma'}) \cup \partial G_{\sigma'} \hookrightarrow P_n(G_\sigma) \cup \partial G_\sigma$.
- Since there are only finitely many hyperbolic groups involved, choose $n \geq 0$ such that all the previously defined Rips complexes are contractible.

Since all the twisting elements are trivial, it follows that $EG(\mathcal{Y})$ is an $E\mathcal{Z}$-complex of classifying spaces compatible with $G(\mathcal{Y})$. □

From now on, $G(\mathcal{Y})$ is a complex of groups over a simplicial complex Y satisfying the conditions of V.0.21. We will denote by G the fundamental group of $G(\mathcal{Y})$ and by X its universal covering

Lemma V.1.2. *The $E\mathcal{Z}$-complex of classifying spaces $EG(\mathcal{Y})$ satisfies the limit set property IV.2.4.*

Proof. For every pair of simplices $\sigma \subset \sigma'$ of Y, $G_{\sigma'}$ is a quasiconvex subgroup of G_σ, so the map $\phi_{\sigma,\sigma'} : \partial G_{\sigma'} \to \partial G_\sigma$ realises a $G_{\sigma'}$-equivariant homeomorphism $\partial G_{\sigma'} \to \Lambda G_{\sigma'} \subset \partial G_\sigma$ by a result of Bowditch [6].

For every simplex σ of Y, the family

$$\mathcal{F}_\sigma = \left\{ \bigcap_{i=1}^{n} g_i G_{\sigma_i} g_i^{-1} | \ g_0, \ldots, g_n \in G_\sigma, \sigma_1, \ldots, \sigma_n \in \mathrm{st}(\sigma), n \in \mathbb{N} \right\}$$

is contained in the family of quasiconvex subgroups of G_σ. Indeed, let $g_0, \ldots, g_n$ be elements of G. Then, as X is CAT(0), $\bigcap_{0 \leq i \leq n} g_i G_\sigma g_i^{-1} = \bigcap_{v \in \Gamma} g_i G_v g_i^{-1}$, where Γ is a graph containing all the vertices of the simplices $g_0\sigma, \ldots, g_n\sigma$ and contained in the convex hull of the $g_0\sigma, \ldots, g_n\sigma$. For such subgroups, the equality $\Lambda H_1 \cap \Lambda H_2 = \Lambda(H_1 \cap H_2)$ holds by Lemma I.3.29. $\square$

Lemma V.1.3. *The $E\mathcal{Z}$-complex of classifying spaces $EG(\mathcal{Y})$ satisfies the convergence property IV.3.8.*

Proof. This is Proposition 1.8 of [14]. $\square$

Lemma V.1.4. *The $E\mathcal{Z}$-complex of classifying spaces $EG(\mathcal{Y})$ satisfies the finite height property IV.2.5.*

Proof. A quasiconvex subgroup of a hyperbolic group has finite height by a result of [22]. $\square$

Theorem IV.0.4 now implies the following:

Corollary V.1.5. *The fundamental group of $G(\mathcal{Y})$ admits a classifying space for proper actions and an $E\mathcal{Z}$-structure.* $\square$

Note that this corollary does not use the hyperbolicity of X.

V.2 A combination theorem.

Let $(\overline{EG}, \partial G)$ be the $E\mathcal{Z}$-structure constructed in the previous section. We now prove that G is a hyperbolic group, by proving that it is a uniform convergence group on its boundary.

So far, the topology on $\overline{EG}$ and ∂G was defined by choosing a specific, although arbitrary, basepoint. In forthcoming proofs, we will choose neighbourhoods centred at points which are relevant to the geometry of the problem.

Definition V.2.1. Let $\delta \geq 0$ be such that the space X is δ-hyperbolic. We denote by $\langle ., . \rangle$ the Gromov product on X and an extension to $\overline{X}$. For $z \in \overline{X}$, $k \geq 0$ and $x_0 \in X$ a basepoint, let

$$W_k(z) = \left\{ x \in \overline{X} \text{ such that } \langle x, z \rangle_{x_0} \geq k \right\}.$$

For $\eta \in \partial X$ and $k \geq 0$, the family of subsets $(W_k(\eta))$ forms a basis of (not necessarily open) neighbourhoods of η in $\overline{X}$.

Recall that $d_{\max}$ was defined in IV.3.3 as a constant such that domains of points of $\partial_{Stab}G$ have at most $d_{\max}$ simplices and a geodesic segment contained in the open simplicial neighbourhood of the domain of a point of $\partial_{Stab}G$ meets at most $d_{\max}$ simplices.

We also give the following useful lemma.

Lemma V.2.2. *Let Γ be a finite connected graph contained in the 1-skeleton of X, and $\Gamma' \subset \Gamma$ a connected subgraph. Then $\cap_{v\in\Gamma} G_v$ is hyperbolic and quasiconvex in $\cap_{v\in\Gamma'} G_v$.*

Proof. This follows from an easy induction on the number of vertices of Γ, together with Lemma I.3.29. □

Lemma V.2.3. *Let (g_n) be an injective sequence of elements of G, and suppose there exist vertices v_0 and v_1 of X such that $g_n v_0 = v_1$ for infinitely many n. Then there exist $\xi_+, \xi_- \in \partial G$ and a subsequence $(g_{\varphi(n)})$ such that for every compact subset K of $\partial G \setminus \{\xi_-\}$, the sequence of translates $g_{\varphi(n)} K$ uniformly converges to ξ_+.*

Proof. It is enough to prove the result when $g_n v_0 = v_0$ for infinitely many n. Since G_{v_0} is hyperbolic, we can assume that there exists a subsequence of (g_n), that we still denote (g_n), and points $\xi_+, \xi_- \in \partial G_{v_0}$ such that for every compact subset K of $\overline{EG_{v_0}} \setminus \{\xi_-\}$, the sequence of translates $g_n K$ uniformly converges to ξ_+. Throughout this proof, we choose v_0 as the basepoint.

Let σ be a simplex of X containing v_0.

If σ is not contained in $D(\xi_-)$, then the convergence property IV.3.8 implies that, up to a subsequence, we can assume that the sequence of $g_n \partial G_\sigma$ uniformly converges to ξ_+ in ∂G_{v_0}.

If σ is contained in $D(\xi_-)$, then the subset $\partial G_\sigma \subset \partial G_{v_0}$ consists of at least two points among which there is ξ_-. Since for any other point α of ∂G_σ we have that $g_n \alpha$ tends to ξ_+, the convergence property IV.3.8 implies that one of the following situations happens:

- $g_n G_\sigma$ only takes finitely many values of cosets, in which case we can find a subsequence (g_n) such that $g_n \partial G_\sigma$ is constant and contains ξ_+. This means that we can write $g_n = g'_n . g$ where g is in the stabiliser of v_0 and g'_n in a sequence in the stabiliser of σ. Up to replacing g_n by g'_n, we can assume that g_n fixes σ.

- $g_n G_\sigma$ takes infinitely many values of cosets, in which case we can find a subsequence (g_n) such that $g_n \partial G_\sigma$ uniformly converges to ξ_+.

As domains are finite subcomplexes of X by Proposition IV.3.2, we can iterate this procedure a finite number of times so as to obtain a subsequence (g_n) and a subcomplex $F \subset D(\xi_-) \cap D(\xi_+)$ such that

- F is fixed pointwise under all the g_n,

- for every simplex σ in $(st(F) \setminus F)$ and every vertex v of $\sigma \cap F$, we have that $g_n \partial G_\sigma$ uniformly converges to ξ_+ in ∂G_v.

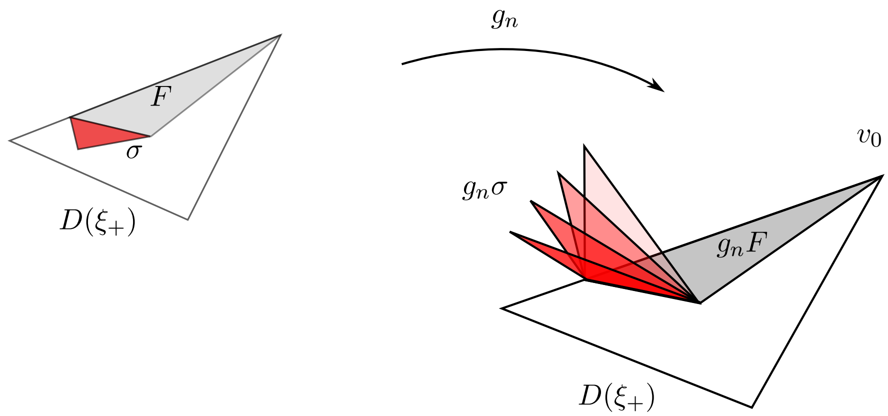

Figure V.1.

For every vertex v of $D(\xi_-)$, choose U_v to be a neighbourhood of ξ_- in ∂G_{v_0}. Choose a ξ_--family $\mathcal{U}'$ which is nested in $\{U_v, v \in D(\xi_-)\}$, and choose $\varepsilon \in (0,1)$. We can further assume that for every simplex σ of F and every vertex v of σ, the subset $\overline{EG_\sigma} \setminus U'_v$ is infinite. Let $K = \partial G \setminus V_{\mathcal{U}',\varepsilon}(\xi_-)$.

We now prove that, up to a subsequence, the sequence of translates g_nK uniformly converges to ξ_+. Because of the definition of neighbourhoods of points of $\partial_{Stab}G$, we need to treat different cases.

Let σ be a simplex of F containing v_0, so that $G_\sigma \subset G_{v_0}$, and v a vertex of σ distinct from v_0. Since G_v is hyperbolic, there exists a subsequence of (g_n), that we still denote (g_n), and points $\xi'_+, \xi'_- \in \partial G_v$ such that for every compact subset K' of $\overline{EG_v} \setminus \{\xi'_-\}$, the sequence of translates g_nK' uniformly converges to ξ'_+. By definition of ξ_+ and ξ_-, we already have that the sequence $g_n(\overline{EG_{v_0}} \setminus U'_{v_0})$ uniformly converges to ξ_+ in ∂G_{v_0}. We thus have that $g_n(\overline{EG_\sigma} \setminus U'_{v_0})$ uniformly converges to ξ_+ in ∂G_v. Since $\overline{EG_\sigma} \setminus U'_{v_0}$ is infinite by construction, this implies that $\xi'_+ = \xi_+$. If we had $\xi'_- \neq \xi_-$, then $g_n\overline{EG_\sigma}$ would uniformly converge to ξ_+, contradicting the fact that $g_n\overline{EG_\sigma} = \overline{EG_\sigma}$ since g_n fixes σ. Therefore $\xi'_- = \xi_-$. This implies that $g_n(\partial G_v \setminus U'_v)$ uniformly converges to ξ_+ in ∂G_v. Since F is finite, an easy induction shows that there exists a subsequence, still denoted (g_n), such that $g_n(\partial G_v \setminus U'_v)$ uniformly converges to ξ_+ in ∂G_v for every vertex v of F.

Let $\widetilde{x} \in K$, and $x \in \bar{p}(\widetilde{x}) \setminus F$. Let σ be the first simplex touched by $[v_0, x]$ after leaving F. It follows from the definition of F that the sequence of simplices $(g_n\sigma)$ is such that for some (hence any) vertex v of $\sigma \cap F$, the sequence of $(\partial G_{g_n\sigma})$ uniformly converges to ξ_+ in ∂G_v. It follows from the convergence criterion IV.6.16 that the sequence $(g_n\widetilde{x})$ converges to ξ_+. Since $\widetilde{x} \notin V_{\mathcal{U},\varepsilon}(\xi_-)$, we have $\partial G_\sigma \not\subset U_v$ for some (hence any) vertex v of F. Since $\mathcal{U}'$ is nested in $\{U_w, w \in V(\xi_-)\}$, it follows that

$$\partial G_\sigma \cap U'_v = \varnothing,$$

this being true for every $\widetilde{x} \in K$ and $x \in \bar{p}(\widetilde{x}) \setminus F$. We already have that for every vertex v of F, the sequence of $g_n.(\partial G_v \setminus U_v)$ uniformly converges to ξ_+ by the above discussion.

As F is a finite subcomplex of X, the convergence criterion IV.6.16 now shows that the sequence $(g_n.K)$ uniformly converges to ξ_+. □

Lemma V.2.4. *Let (g_n) be an injective sequence of elements of G. Suppose that for some (hence any) vertex v the sequence $(g_n v)$ is bounded, but there do not exist vertices v_0 and v_1 of X such that $g_n v_0 = v_1$ for infinitely many n. Then there exist $\xi_+, \xi_- \in \partial G$ and a subsequence $(g_{\varphi(n)})$ such that for every compact subset K of $\partial G \setminus \{\xi_-\}$, the sequence of translates $g_{\varphi(n)} K$ uniformly converges to ξ_+.*

Proof. Choose an arbitrary vertex v and a point $\widetilde{x}$ of EG_v. As ∂G is compact by Theorem IV.6.13 and $(g_n v)$ is bounded, we can choose a subsequence, still denoted (g_n), and points $\xi_+, \xi_- \in \partial_{Stab} G$ such that $g_n \widetilde{x} \to \xi_+$ and $g_n^{-1} \widetilde{x} \to \xi_-$. We choose a vertex v_0 of $D(\xi_+)$ to be the basepoint, and let $\widetilde{x_0} \in EG_{v_0}$. By Theorem IV.7.1, we still have $g_n \widetilde{x_0} \to \xi_+$ and $g_n^{-1} \widetilde{x_0} \to \xi_-$.

Claim 1:

- For every $\eta \in \partial X$, the geodesic ray $[g_n v_0, g_n \eta)$ does not meet $D(\xi_+)$ for n large enough.
- For every $\xi \in \partial_{Stab} G$, the subset $\mathrm{Geod}(g_n v_0, g_n D(\xi))$ does not meet $D(\xi_+)$ for n large enough.

Let $z \in \partial G$. If $z \in \partial X$, we denote by $D(z)$ the singleton $\{z\}$. By contradiction, suppose that there exists an infinite number of n for which there exist $y_n \in D(\xi_+)$ and $x_n \in \mathrm{Geod}(v_0, D(z))$ such that $g_n x_n = y_n$. As (y_n) is bounded by Proposition IV.3.2, the assumption on (g_n) implies that (x_n) is bounded too. Since x_n lies on $\mathrm{Geod}(v_0, D(z))$ for every n, the containment lemma IV.1.3 and the finiteness lemma IV.1.5 imply that, up to a subsequence, we can assume that x_n always lies in the same simplex σ of X. Furthermore, since $D(\xi_+)$ is finite by Proposition IV.3.2, we can assume, up to a subsequence, that y_n lies in a simplex σ' of X for every n. As the action of G on X is without inversion, this implies that $g_n \sigma = \sigma'$ for every n, which was excluded by assumption.

Claim 2: For every ξ in ∂G, the sequence $g_n \xi$ converges to ξ_+.

Let $\mathcal{U}$ be a ξ_+-family, $\mathcal{U}'$ a ξ_+-family that is $3d_{\max}$-nested in $\mathcal{U}$ and $\varepsilon > 0$. Recall that, by assumption on (g_n), the vertex $g_n v_0$ does not belong to $D(\xi_+)$ for n big enough. Furthermore, since $g_n \widetilde{x_0} \to \xi_+$, we have that $\overline{EG}_{\sigma_{\xi_+,\varepsilon}(g_n v_0)} \subset U'_v$ for n large enough and for some (hence every) vertex v of $D(\xi_+) \cap \sigma_{\xi_+,\varepsilon}(g_n v_0)$. We split the proof of the claim in two cases.

Let $\eta \in \partial X$. For n large enough, the path $[g_n v_0, g_n \eta)$ does not meet $D(\xi_+)$ by Claim 1. By Proposition IV.3.2 ,we can choose $y \in D(\xi_+)$ which minimises the distance to

$\text{Geod}(g_n v_0, g_n\eta)$. Let τ (resp. τ') be a simplex of $\text{N}(D(\xi_+)) \setminus D(\xi_+)$ whose interior is crossed by $[y, g_n v_0]$ (resp. $[y, g_n\eta))$ at a point u (resp. u'). By convexity of the function $z \mapsto d(z, [g_n v_0, g_n\eta))$, it follows from the definition of y that the geodesic segment $[u, u']$ does not meet $D(\xi_+)$, thus yielding a path of simplices of length at most $d_{\max}$ between τ and τ' in $\text{N}(D(\xi_+)) \setminus D(\xi_+)$. Lemma IV.1.7 implies that there exists a path of simplices of length at most $d_{\max}$ between τ and the exit simplex $\sigma_{\xi_+,\varepsilon}(g_n v_0)$ (resp. between τ' and the exit simplex $\sigma_{\xi_+,\varepsilon}(g_n\eta)$) in $\text{N}(D(\xi_+)) \setminus D(\xi_+)$. Thus for n large enough, there is a path of simplices of length at most $3d_{\max}$ from $\sigma_{\xi_+,\varepsilon}(g_n v_0)$ to $\sigma_{\xi_+,\varepsilon}(g_n\eta)$ in $\text{N}(D(\xi_+)) \setminus D(\xi_+)$. As $\overline{EG}_{\sigma_{\xi_+,\varepsilon}(g_n v_0)} \subset U'_v$ for n large enough and for some (hence every) vertex v of $D(\xi_+) \cap \sigma_{\xi_+,\varepsilon}(g_n v_0)$, it follows from the fact that $\mathcal{U}'$ is $3d_{\max}$-nested in $\mathcal{U}$ that $\overline{EG}_{\sigma_{\xi_+,\varepsilon}(g_n\eta)} \subset U_v$ for n large enough and for some (hence every) vertex v of $D(\xi_+) \cap \sigma_{\xi_+,\varepsilon}(g_n\eta)$. It thus follows that $(g_n\eta)$ converges to ξ_+.

Let $\xi \in \partial_{Stab}G$. For n large enough, $\text{Geod}(g_n v_0, g_n D(\xi))$ does not meet $D(\xi_+)$ by Claim 1. Let $x \in D(\xi)$ and, by Proposition IV.3.2, let y be a point of $D(\xi_+)$ which minimises the distance to $\text{Geod}(g_n v_0, g_n x)$. Using the same reasoning as above, we get, for n large enough, a path of simplices of length at most $3d_{\max}$ from $\sigma_{\xi_+,\varepsilon}(g_n v_0)$ to $\sigma_{\xi_+,\varepsilon}(g_n x)$ in $\text{N}(D(\xi_+)) \setminus D(\xi_+)$. As $\overline{EG}_{\sigma_{\xi_+,\varepsilon}(g_n v_0)} \subset U'_v$ for n large enough and for some (hence every) vertex v of $D(\xi_+) \cap \sigma_{\xi_+,\varepsilon}(g_n v_0)$, it follows from the fact that $\mathcal{U}'$ is $3d_{\max}$-nested in $\mathcal{U}$ that, for n large enough and for every $x \in D(\xi)$, $\overline{EG}_{\sigma_{\xi_+,\varepsilon}(g_n x)} \subset U_v$ for some (hence every) vertex v of $D(\xi_+) \cap \sigma_{\xi_+,\varepsilon}(g_n x)$. It thus follows that $(g_n\xi)$ converges to ξ_+.

In the same way, we prove that for every $\xi \in \partial G$, the sequence $g_n^{-1}\xi$ converges to ξ_-. To conclude the proof of the lemma, it remains to show that this convergence can be made uniform away from ξ_-:

Claim 3: For every $\xi \neq \xi_-$ in ∂G, there is a subsequence (g_n) and a neighbourhood U of ξ in ∂G such that the sequence of $g_n U$ uniformly converges to ξ_+.

Once again, we split the proof in two cases.

Let $\xi \in \partial_{Stab}G$. We already have that $g_n\xi \to \xi_+$ by Claim 2. In order to find a ξ-family $\mathcal{U}$ and a constant ε such $g_n V_{\mathcal{U},\varepsilon}(\xi)$ uniformly converges to ξ_+, it is enough, using the same reasoning as in Claim 2, to find a ξ-family $\mathcal{U}$ and a constant ε such that for every x in $D(\xi) \cup \text{Cone}_{\mathcal{U},\varepsilon}(\xi)$, the geodesic from $g_n v_0$ to $g_n x$ does not meet $D(\xi_+)$.
By Claim 1, we already have that for n large enough, no geodesic from $g_n v_0$ to a point of $g_n D(\xi)$ meets $D(\xi_+)$. As $\xi \neq \xi_-$, we choose a ξ-family $\mathcal{U}$, a ξ_--family $\mathcal{U}'$ and constants $\varepsilon, \varepsilon' \in (0, 1)$ such that the neighbourhoods $V_{\mathcal{U},\varepsilon}(\xi)$ and $V_{\mathcal{U}',\varepsilon'}(\xi_-)$ are disjoint. Up to a subsequence, we have by the first claim that $g_n D(\xi)$ does not meet $D(\xi_+)$. It now follows from the definition of $\mathcal{U}$ and the fact that $g_n^{-1}\xi_+ \to \xi_-$ that $\text{Cone}_{\mathcal{U},\varepsilon}(\xi)$ does not meet the sets $g_n^{-1}D(\xi_+)$, hence the sets $g_n\text{Cone}_{\mathcal{U},\varepsilon}(\xi)$ do not meet $D(\xi_+)$. Now this implies that for every x in $\text{Cone}_{\mathcal{U},\varepsilon}(\xi)$, the geodesic from $g_n v_0$ to $g_n x$ does not meet $D(\xi_+)$: indeed, this geodesic must meet $g_n D(\xi)$ since the geodesic from v_0 to a point of $\text{Cone}_{\mathcal{U},\varepsilon}(\xi)$ must

meet $D(\xi)$, and we already proved that a geodesic segment from $g_n v_0$ to a point of $g_n D(\xi)$ does not meet $D(\xi_+)$. Now the same proof as in Claim 2 shows that $g_n V_{\mathcal{U},\varepsilon}(\xi)$ uniformly converges to ξ_+.

Let $\eta \in \partial X$. We already know that $g_n \eta \to \xi_+$ by Claim 2. In order to find a neighbourhood U of η in $\overline{X}$ such that such $g_n V_U(\eta)$ uniformly converges to ξ_+, it is enough, using the same reasoning as in Claim 2, to find a neighbourhood U of η in $\overline{X}$ such that for every x in U, the geodesic from $g_n v_0$ to $g_n x$ does not meet $D(\xi_+)$.
First, notice that the distance from the geodesic rays $[g_n v_0, g_n \eta)$ to $D(\xi_+)$ is uniformly bounded below: indeed, if this was not the case, the same reasoning as in Claim 1 would imply the existence of simplices σ, σ' of X such that $g_n \sigma \cap \sigma' \neq \varnothing$. This in turn would imply that, up to a subsequence, there exist subsimplices $\tau \subset \sigma$ and $\tau' \subset \sigma'$ such that $g_n \tau = \tau'$, which was excluded. Thus, let $\varepsilon > 0$ be such a uniform bound. Let also

$$M = \sup_{x \in D(\xi_+), n \geq 0} d(g_n v_0, x).$$

Now consider the neighbourhood $V_{M,\varepsilon}(\eta)$ of η in $\overline{X}$. Let $x \in X$ be in that neighbourhood, and let γ be a parametrisation of the geodesic from v_0 to x. Suppose by contradiction that the geodesic from $g_n v_0$ to $g_n x$ does meet $D(\xi_+)$. Then, by definition of M, the geodesic segment $g_n \gamma\big([0, M]\big)$ meets $D(\xi_+)$. But as this geodesic segment is in the open ε-neighbourhood of $[g_n v_0, g_n \eta)$, we get our contradiction from the definition of ε.
Thus for every x in $V_{M,\varepsilon}(\eta)$, the geodesic from $g_n v_0$ to $g_n x$ does not meet $D(\xi_+)$, and we are done. □

Lemma V.2.5. *Let (g_n) be an injective sequence of elements of G, and suppose that for some (hence every) vertex v_0 of X, $d(v_0, g_n v_0) \to \infty$. Since $(\overline{EG}, \partial G)$ is an $E\mathcal{Z}$-structure for G by Theorem IV.7.1, we can assume up to a subsequence that there exist $\xi_+, \xi_- \in \partial G$ such that for every compact subset $K \subset EG$, we have $g_n K \to \xi_+$ and $g_n^{-1} K \to \xi_-$. Then there exists a subsequence $(g_{\varphi(n)})$ such that for every compact subset K of $\partial G \setminus \{\xi_-\}$, the sequence of translates $g_{\varphi(n)} K$ uniformly converges to ξ_+.*

Proof. If $\xi_- \in \partial X$, let U be a neighbourhood of ξ_- in ∂X and $K = \partial G \setminus V_U(\xi_-)$. Since X has finitely many isometry types of simplices, it follows from Lemma IV.5.3 that we can choose a subneighbourhood U' of U containing ξ_- and such that any path from $U' \cap X$ to $X \setminus U$ meets at least $d_{\max}$ simplices.

If $\xi_- \in \partial_{Stab} G$, let $\mathcal{U}$ be a ξ_--family, and $\varepsilon \in (0,1)$, and let $K = \partial G \setminus V_{\mathcal{U},\varepsilon}(\xi_-)$. We also choose another ξ_--family $\mathcal{U}'$ which is $2d_{\max}$-refined in $\mathcal{U}$.
We want to prove that $(g_n K)$ uniformly converges to ξ_+. Recall that the sets $W_k(g_n v_0)$ were defined in V.2.1.

Claim 1: For every k, the following holds:

- If $\xi_- \in \partial X$, we have $g_n\big(\overline{X} \setminus U'\big) \subset W_k(g_n v_0)$ for n large enough.

- If $\xi_- \in \partial_{Stab}G$, we have $g_n\big(\overline{X} \setminus \widetilde{\text{Cone}}_{\mathcal{U}',\varepsilon}(\xi_-)\big) \subset W_k(g_n v_0)$ for n large enough.

We split the proof in two cases.

Suppose that $\xi_- \in \partial X$. First notice that since $g_n^{-1}v_0 \to \xi_-$, there exists a constant C such that for every $n \geq 0$ and every $x \notin U$, we have $\langle g_n^{-1}v_0, x\rangle_{v_0} \leq C$. Since we also have $d(g_n^{-1}v_0, v_0) \to \infty$, the claim follows.

Suppose now that $\xi_- \in \partial_{Stab}G$. We start by proving by contradiction that there exists a constant C such that for every $n \geq 0$ and every $x \notin \widetilde{\text{Cone}}_{\mathcal{U}',\varepsilon}(\xi_-)$, we have $\langle g_n^{-1}v_0, x\rangle_{v_0} \leq C$. The containment lemma IV.1.3 yields a constant m such that a geodesic path of length at most δ meets at most m simplices, where δ is the hyperbolicity constant of X. Let $\mathcal{U}''$ be a ξ_--family that is m-nested in $\mathcal{U}'$. Since we are reasoning by contradiction, then, up to a subsequence, there exist points $y_n \notin \widetilde{\text{Cone}}_{\mathcal{U}',\varepsilon}(\xi_-)$ such that $\langle g_n^{-1}v_0, y_n\rangle_{v_0} \to \infty$. By hyperbolicity of X , the geodesic segments $[v_0, g_n^{-1}v_0]$ and $[v_0, y_n]$ stay δ-close until time $\langle g_n^{-1}v_0, y_n\rangle_{v_0} \to \infty$. Moreover, as $g_n^{-1}\widetilde{x_0} \to \xi_-$ for any point $\widetilde{x_0} \in EG_{v_0}$, we have $g_n^{-1}v_0 \in \text{Cone}_{\mathcal{U}'',\varepsilon}(\xi_-)$ for n large enough. Thus, for n large enough, there exist points $a_n \in [v_0, y_n]$ and $b_n \in [v_0, g_n^{-1}v_0] \cap \text{Cone}_{\mathcal{U}'',\varepsilon}(\xi_-)$ and a path of simplices of length at most m between a_n and b_n which is contained in $X \setminus D(\xi_-)$ (see Figure V.2).

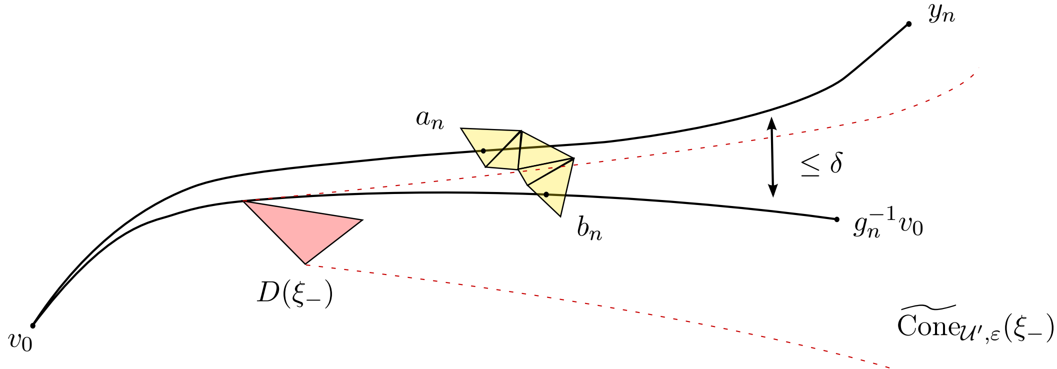

Figure V.2.

The refinement lemma IV.4.10 now implies that a_n and y_n both are in $\text{Cone}_{\mathcal{U}',\varepsilon}(\xi_-)$ for n large enough, a contradiction.
Now the same reasoning as before shows that for every $k \geq 0$, there exists N such that for every $n \geq N$ and every $x \notin \widetilde{\text{Cone}}_{\mathcal{U}',\varepsilon}(\xi_-)$, $\langle v_0, x\rangle_{g_n^{-1}v_0} \geq k$, hence $\langle g_n v_0, g_n x\rangle_{v_0} \geq k$.

Claim 2: For every k, we have $g_n\bar{p}(K) \subset W_k(g_n v_0)$ for n large enough.

Suppose that $\xi_- \in \partial X$. By definition of U', we have that for every $z \in K$, $\bar{p}(z) \cap U' = \varnothing$. Thus $\bar{p}(K) \subset \overline{X} \setminus U'$ and the result follows from Claim 1.

Suppose now that $\xi_- \in \partial_{Stab}G$, and let $z \in K$. Suppose by contradiction that $\bar{p}(z) \cap \widetilde{\text{Cone}}_{\mathcal{U}',\varepsilon}(\xi_-) \neq \varnothing$. If $\bar{p}(z)$ is contained in $X \setminus D(\xi_-)$, then the refinement lemma

IV.4.10 implies that $\bar{p}(z) \subset \widetilde{\mathrm{Cone}}_{\mathcal{U},\varepsilon}(\xi_-)$; hence $z \in V_{\mathcal{U},\varepsilon}(\xi_-)$, which is absurd. If $\bar{p}(z)$ meets $D(\xi_-)$, then since $\mathcal{U}'$ is $2d_{\max}$-refined in $\mathcal{U}$ it follows from the refinement lemma IV.4.10 and Lemma IV.5.20 that $z \in V_{\mathcal{U},\varepsilon}(\xi_-)$, a contradiction. Thus $\bar{p}(K) \subset \overline{X} \setminus \widetilde{\mathrm{Cone}}_{\mathcal{U}',\varepsilon}(\xi_-)$ and the result follows from Claim 1.

Claim 3: g_nK uniformly converges to ξ_+.

Once again, we split the proof in two cases.

Suppose that $\xi_+ \in \partial X$. Then, as $g_nv_0 \to \xi_+$, it follows from Claim 2 that for every k, $g_n\bar{p}(K) \subset W_k(\xi_+)$ for n large enough. It then follows from the convergence criterion IV.6.16 that g_nK uniformly converges to ξ_+.

Suppose now that $\xi_+ \in \partial_{Stab}G$. Let $\mathcal{U}_+$ be a ξ_+-family and $\varepsilon \in (0,1)$. Since X is δ-hyperbolic, let m be a constant such that a geodesic path of length at most δ meets at most m simplices, and let $\mathcal{U}'_+$ be a ξ_+-family that is m-nested in $\mathcal{U}_+$. As $g_n\widetilde{x_0} \to \xi_+$ for any $\widetilde{x_0} \in EG_{v_0}$, we have $g_nv_0 \in \mathrm{Cone}_{\mathcal{U}'_+,\varepsilon}(\xi_+)$ for n large enough. For every $T \geq 0$, we can choose n large enough so that the geodesic segments $[v_0, g_nv_0]$ and $[v_0, g_nx]$, $x \in \bar{p}(K)$, remain δ-close up to time T (if we choose k large enough in Claim 2). In particular, we can choose k and N large enough so that, for every $n \geq N$ and every $x \in \bar{p}(K)$, there exists a path of simplices of length at most m in $X \setminus D(\xi_+)$ between a point of $[v_0, g_nv_0] \cap \mathrm{Cone}_{\mathcal{U}'_+,\varepsilon}(\xi_+)$ and a point of $[v_0, g_nx]$. The refinement lemma IV.4.10 now implies that $g_n\bar{p}(K) \subset \widetilde{\mathrm{Cone}}_{\mathcal{U}_+,\varepsilon}(\xi_+)$ for $n \geq N$, hence $g_nK \subset V_{\mathcal{U}_+,\varepsilon}(\xi_+)$ for $n \geq N$. Thus, g_nK uniformly converges to ξ_+. □

Corollary V.2.6. *The group G is a convergence group on ∂G.*

Proof. This follows from Lemma V.2.3, Lemma V.2.4 and Lemma V.2.5. □

To prove that G is hyperbolic, it remains to show that every point of ∂G is conical.

Lemma V.2.7. *Every point of ∂G is a conical limit point for ∂G.*

Proof. Consider first a point in ∂G_v for some vertex v of X. It is a conical limit point for G_v on ∂G_v, since G_v is hyperbolic. Therefore it is a conical point for G_v on ∂G, hence for G since G is a convergence group on ∂G by Corollary V.2.6.

Now consider a point $\eta \in \partial X$. Since the action of G on X is cocompact, we can find a sequence (g_n) of elements of G and a simplex σ such that $(g_n\sigma)$ uniformly converges to η in $\overline{X}$ and such that for every n, the geodesic ray $[v_0, \eta)$ meets the interior of $g_n\sigma$. Let v be a vertex of σ and $\widetilde{x} \in EG_v$.

Claim: Up to multiplying each g_n on the right by an element of G_v and taking a subsequence, we can further assume that $g_n^{-1}\widetilde{x}$ converges to a point $\xi_- \in \partial G \setminus \partial G_v$.

Consider the first simplex touched by the geodesic $[v, g_n^{-1}v]$ after leaving v. Since the action of G on X is cocompact, we can assume up to a subsequence that this sequence of simplices is in the same G-orbit. Now up to multiplying each g_n by an element of G_v, we can further assume that this sequence of simplices is constant at a unique simplex σ_1. Up to a subsequence, we can further assume that all the geodesic segments $[v, g_n^{-1}v]$ leave σ_1 along the same open simplex τ_1. Now consider the simplex $\sigma_2^{(n)}$ touched by $[v, g_n^{-1}v]$ after leaving τ_1 and choose a G_{σ_1}-orbit in EG_{τ_1}. Since G_{σ_1} is quasiconvex in G_{τ_1}, this orbit is a quasiconvex subset Q_1 of EG_{τ_1}; choose a basepoint y of Q_1. For every n, choose a point $x_n \in EG_{\sigma_2^{(n)}}$ and let y_n be a projection of x_n on the quasiconvex subset Q_1. By definition of Q_1, there exists an element $h_n \in G_{\sigma_1} \subset G_v$ such that $h_n y_n = y$. This implies that for every n, the subset $h_n EG_{\sigma_2^{(n)}}$ contains a point that projects to y. In particular, no subsequence of $h_n \overline{EG}_{\sigma_2^{(n)}}$ converges to a point of ∂G_{σ_1}. Suppose by contradiction that there exists a subsequence of $h_n \overline{EG}_{\sigma_2^{(n)}}$ which converges to a point $z \in \partial G_{\tau_1}$. Since G_{τ_1} is a convergence group on $\overline{EG}_{\tau_1}$ by Proposition I.3.20, it follows that for every $x \in \overline{EG}_{\tau_1}$ except maybe one point, $h_n x$ converges to z. But as Q_1 is stable under all the h_n, such a z belongs to $\partial G_{\sigma_1} \subset \partial G_{\tau_1}$, contradicting the fact that no subsequence of $h_n \overline{EG}_{\sigma_2^{(n)}}$ converges to a point of ∂G_{σ_1}. Thus, no subsequence of $h_n \overline{EG}_{\sigma_2^{(n)}}$ converges to a point of ∂G_{τ_1} and the convergence property IV.3.8 now implies that, up to a subsequence, we can assume that $h_n \overline{EG}_{\sigma_2^{(n)}}$ is constant. Up to a subsequence, we can further assume that $\sigma_2^{(n)}$ is constant at σ_2 and every geodesic segment $[v, g_n^{-1}v]$ leaves σ_2 along the same open simplex τ_2. In view of the above, we replace the sequence (g_n) by $(g_n h_n^{-1})$. Now one of the following happens:

- (i) Suppose that $G_{\sigma_1} \cap G_{\sigma_2}$ is finite. By applying the same reasoning as in the proof of the compactness lemmas IV.6.14 and IV.6.15, either there exists a subsequence of (g_n) such that $g_n^{-1}\widetilde{x}$ converges to a point of ∂X and we are done, or the path of simplices σ_1, σ_2 extends to a path of simplices $\sigma_1, \ldots, \sigma_m$ which are crossed by every geodesic segment $[v, g_n^{-1}v]$ and $g_n^{-1}\widetilde{x}$ converges to a point $\xi_- \in \partial G_{\sigma_m}$. As $D(\xi_-)$ is convex by Proposition IV.3.2 and $G_{\sigma_1} \cap G_{\sigma_2}$ is finite, it follows from Lemma IV.3.7 that $\xi_- \notin \partial G_v$ and we are done.

- (ii) Suppose that $G_{\sigma_1} \cap G_{\sigma_2}$ is infinite. Let $\sigma_3^{(n)}$ be the simplex touched by $[v, g_n^{-1}v]$ after leaving τ_2, and let Q_2 be a $G_{\sigma_1} \cap G_{\sigma_2}$-orbit in EG_{τ_2}. Note that Q_2 is quasiconvex in EG_{τ_2} by Lemma V.2.2. We are thus back to the previous situation with EG_{τ_2} instead of EG_{τ_1}, $\overline{EG}_{\sigma_3^{(n)}}$ instead of $\overline{EG}_{\sigma_2^{(n)}}$ and Q_2 instead of Q_1.

We claim that this procedure eventually stops. Indeed, the containment lemma IV.1.3 yields a constant m such that every geodesic meeting at least m simplices has length at least A, where A is the acylindricity constant. Thus, after at most m applications of this algorithm, we get to situation (i), which concludes the proof of the claim.

By the above discussion, we already have that $g_n^{-1}\widetilde{x} \to \xi_-$ for every $\widetilde{x} \in EG_v$. Thus, by Lemma V.2.5, it is enough, in order to prove Lemma V.2.7, to show that $g_n^{-1}\eta$ does not converge to ξ_-, which we now prove by contradiction.

Suppose $g_n^{-1}\eta$ was converging to ξ_-. For every n, let x_n be a point of $[g_n^{-1}v, g_n^{-1}\eta)$ that is contained in the interior of σ. Since the geodesic ray $[g_n^{-1}v, g_n^{-1}\eta)$ meets σ for every n, the Gromov product $\langle g_n^{-1}v, g_n^{-1}\eta\rangle_v$ is bounded. Thus, ξ_- cannot belong to ∂X, and $\xi_- \in \partial_{Stab}G$.

Now since both $g_n^{-1}\eta$ and $g_n^{-1}\widetilde{x}$ converge to $\xi_- \in \partial_{Stab}G$, both geodesics $[v, g_n^{-1}\eta)$ and $[v, g_n^{-1}v]$ must go through $D(\xi_-)$ for n large enough. But Lemma IV.1.7 and Lemma IV.4.8 imply that for n large enough and any $x \in \sigma$, both geodesic rays $[x, g_n^{-1}\eta)$ and $[x, g_n^{-1}v]$ also meet $D(\xi_-)$. In particular, $[x_n, g_n^{-1}\eta)$ and $[x_n, g_n^{-1}v]$ meet $D(\xi_-)$ for n large enough. As $D(\xi_-)$ is convex by Proposition IV.3.2, this implies that x_n belongs to $D(\xi_-)$, hence so does v, which is absurd by construction of (g_n). □

Corollary V.2.8. *G is a hyperbolic group and ∂G is G-equivariantly homeomorphic to its Gromov boundary.*

Proof. The group G is a convergence group on ∂G by Corollary V.2.6, and every point of ∂G is conical by LemmaV.2.7, thus the result follows from Theorem I.3.22. □

To conclude the proof of Theorem V.0.21, it remains to show that stabilisers embed as quasiconvex subsets.

Proposition V.2.9. *Stabilisers of simplices of X are quasiconvex subgroups of G.*

Proof. It is enough to prove the result for the stabiliser of a vertex v of X. Notice that, by Proposition IV.5.19, the boundary of G_v embeds G_v-equivariantly in ∂G, the latter being G-equivariantly homeomorphic to the Gromov boundary of G by Corollary V.2.8. Hence, the result follows from Theorem I.3.28 □

Proof of Theorem V.0.21: This follows from Corollary V.2.8 and Proposition V.2.9. □

Proof of Corollary V.0.22: This follows from Theorem V.0.21 and Lemma V.1.1. □

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Étant donné un complexe de groupes développable dont les groupes locaux satisfont une propriété donnée, quand peut-on affirmer qu'il en est de même pour son groupe fondamental ? Cette question naturelle de géométrie des groupes a fait l'objet de nombreux travaux dans le cas des graphes de groupes et des complexes de groupes finis. Il existe en revanche peu de résultats concernant les complexes de groupes de dimension supérieure dont les groupes locaux ne sont pas supposés finis. L'objet de cette thèse est de développer des outils géométriques et topologiques pour étudier de tels complexes de groupes, et plus particulièrement les complexes de groupes à courbure négative ou nulle. Étant donné un tel complexe de groupes, on s'intéresse à des propriétés de nature essentiellement asymptotique de son groupe fondamental : existence d'un modèle cocompact d'espace classifiant pour les actions propres, existence d'EZ-structures, hyperbolicité. Ce faisant, on démontre un théorème de combinaison pour les groupes hyperboliques qui généralise aux complexes de groupes de dimension arbitraire un théorème de Bestvina-Feighn.

INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE
UMR 7501
Université de Strasbourg et CNRS
7 Rue René Descartes
67 084 STRASBOURG CEDEX

Tél. 03 68 85 01 29
Fax 03 68 85 03 28

www-irma.u-strasbg.fr
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